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RELAXATION METHODS APPLIED TO DETERMINE THE MOTION, IN TWO DIMENSIONS, OF A VISCOUS FLUID PAST A FIXED CYLINDER

By D. N. DE G. ALLEN (*Imperial College*) and R. V. SOUTHWELL (*Cambridge*)

[Received 13 May 1954]

SUMMARY

In this paper relaxational techniques are applied to the general case of steady laminar motion of an incompressible viscous fluid past a stationary cylinder: that is, to motion at speeds such that neither inertia nor viscosity can be neglected. The governing equation is
$$[u\,\partial/\partial x+v\,\partial/\partial y-\nu\nabla^2]\zeta = 0,$$
where $\nabla^2 \equiv \partial^2/\partial x^2+\partial^2/\partial y^2$, u and v are the component velocities, ν is the kinematic viscosity, and ζ (the vorticity) $= \partial v/\partial x-\partial u/\partial y$. It must be solved in conjunction with the equation of continuity
$$\partial u/\partial x+\partial v/\partial y = 0,$$
which permits the introduction of a stream-function ψ such that
$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x, \qquad \zeta = -\nabla^2\psi.$$
The numerical computations relate to a *circular* cylinder, but the methods are applicable to any shape (an initial conformal transformation changes the independent variables from x and y to α and β, the irrotational velocity-potential and stream-function for flow past the specified cylinder). The flow-patterns (contours of ψ and ζ) change as the 'Reynolds number' R increases; but an introduction of variables involving R makes the change relatively slow, and thereby (e.g.) the accepted solution for $R = 10$ is made a good starting assumption for $R = 100$.

Fig. 6 relates computed values of the total drag with experimental and other theoretical estimates.

1. Introduction. This paper treats two-dimensional (laminar) motion of an incompressible viscous fluid in the most general case—viz. when neither inertia nor viscosity can be neglected. The governing equations are the equation of continuity
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \tag{1}$$
and the dynamical conditions
$$\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right](u,v,w)$$
$$= -\frac{1}{\rho}\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]p+(X,Y,Z)+\nu\nabla^2(u,v,w), \tag{2}$$

in which

$$\left.\begin{array}{l} u,\ v,\ w \text{ are the components of velocity,} \\ X,\ Y,\ Z \text{ are the components of body-force,} \\ p \text{ is the ‘mean normal pressure’,} \\ \rho \text{ is the density and} \\ \nu \text{ is the ‘kinematic viscosity’ of the fluid (\textit{both taken}} \\ \qquad \textit{here as constant}\text{), and} \\ \nabla^2 \text{ stands for } \partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2. \end{array}\right\} \quad (3)$$

In two-dimensional motion $w = Z = 0$ and u, v, X, Y, p are independent of z, so (1) reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad (4)$$

and (2) to

$$\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right](u, v) = -\frac{1}{\rho}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]p+(X, Y)+\nu\nabla^2(u, v), \quad (5)$$

∇^2 now standing for $\qquad \partial^2/\partial x^2+\partial^2/\partial y^2. \quad (6)$

We postulate that the body-forces are conservative so that

$$\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x} = 0, \quad (7)$$

and on that understanding, having regard to (4), we eliminate p from (5) to obtain for *steady* motion (independent of t)

$$[u\,\partial/\partial x+v\,\partial/\partial y]\zeta = \nu\nabla^2\zeta, \quad (8)$$

where ∇^2 has the significance stated in (6) and

$$\zeta \text{ (the vorticity)} = \partial v/\partial x-\partial u/\partial y. \quad (9)$$

2. In relation to plane two-dimensional motion (4), (8), and (9) have to be satisfied in conjunction with appropriate boundary conditions. This paper treats in detail flow past a rigid cylinder (with axis parallel to Oz) of a stream which otherwise (i.e., in the absence of the cylinder) would have uniform velocity U. When, as here, the cylinder is kept stationary, the conditions to be satisfied at its surface are

$$u = v = 0. \quad (10)$$

Other conditions (at infinity) will receive attention later.

3. Even when thus reduced to (4), (8), and (9) the general equations have hitherto proved intractable, except by approximate methods and as relating to fairly small velocities. High-speed solutions have been attained only by the introduction—by L. Prandtl in 1904—of assumptions based on the notion of a ‘boundary layer’ and of a ‘wake’ within which the effects of viscosity are confined; and even so it is necessary to *postulate*

the distribution of the pressure on the cylinder, which strictly should emerge as a result of computation. The only treatment known to us which dispenses with this postulate (**1**) relates to rather slow flow past a circular cylinder.† It has some points of similarity with our relaxational treatment (e.g. its evaluation of the stream-function ψ at discrete nodal points of a square-mesh net).

4. Here, too, flow past a *circular* cylinder is discussed, for the reasons (i) that both in theory and in experiment this shape has received more attention than others and (ii) that it is sufficiently 'bluff' to exemplify difficulties which such shapes may oppose to computation. Actually an initial step in the relaxational treatment makes the shape a matter of relatively small concern: namely, a conformal transformation which changes the independent variables from x and y to α and β, the velocity-potential and stream-function for irrotational steady flow past the given cylinder. Thereby the field of computation is transformed into an infinite plane containing a rectilinear slit, so computation can be effected on square-mesh nets having no 'irregular stars': the shape affects them only in that the transformed equations involve h, the modulus of transformation.

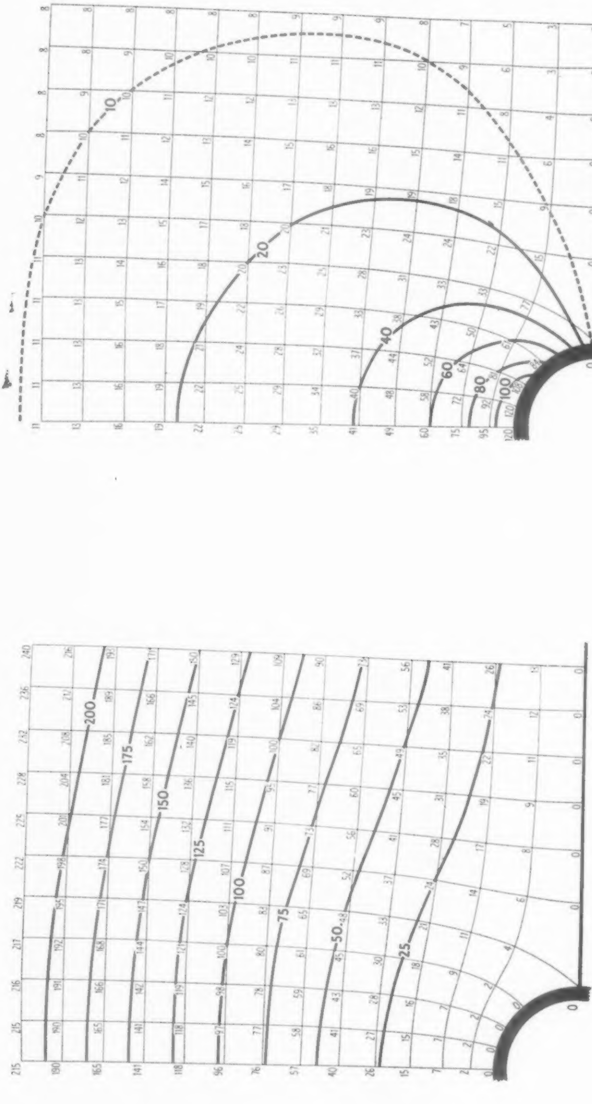
With a view to the numerical computations, all quantities are made 'non-dimensional'; and in consequence U (the velocity at infinity) appears in a numerical parameter

$$R = UL/\nu \tag{11}$$

(the Reynolds number of the motion) conjoined with ν and L, a representative dimension of the inserted cylinder. A further transformation which replaces β by $\beta'' = \beta R^{\frac{1}{2}}$ gives to the equations forms such that a solution found for some particular speed U_1 can be made a starting assumption from which, with relatively little computation, the solution for another speed U_2 can be derived. In sections 14–16 yet another transformation (devised by Allen) is employed to derive equations which, as involving exponentials, are not closely represented by the customary approximations in finite differences.

5. Results are presented in Figs. 1–7, of which their legends provide sufficient explanation. Figs. 1–5 exhibit, by contours of the 'non-dimensional' stream-function (ψ) and vorticity (ζ), the general characteristics of the flow when $R = 0, 1, 10, 10^2, 10^3$. Fig. 6 compares computed values of total 'drag' with experimental and with other theoretical estimates. Fig. 7 shows how the computed pressure-distribution alters with Reynolds number.

† Another treatment, also for fairly slow flow past a circular cylinder, has recently been reported by Kawaguti (**2**).



Values and contours of ψ''

Values and contours of ζ''

(N.B. For $R = 0$, only relative values are significant)

FIG. 1. Stream-function (ψ) and vorticity (ζ'') for Reynolds number $R = 0$.

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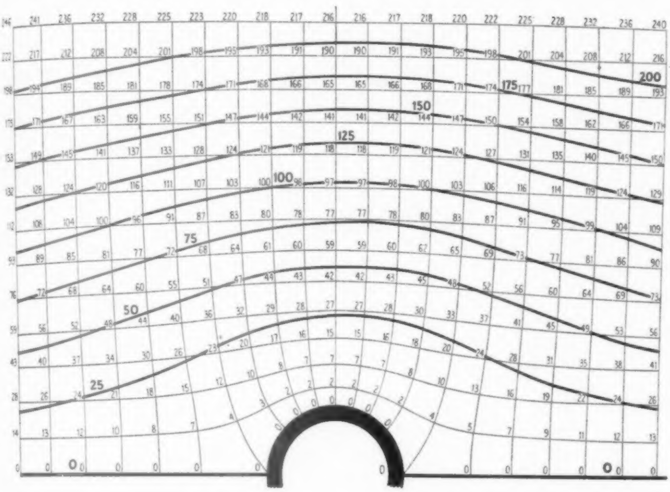
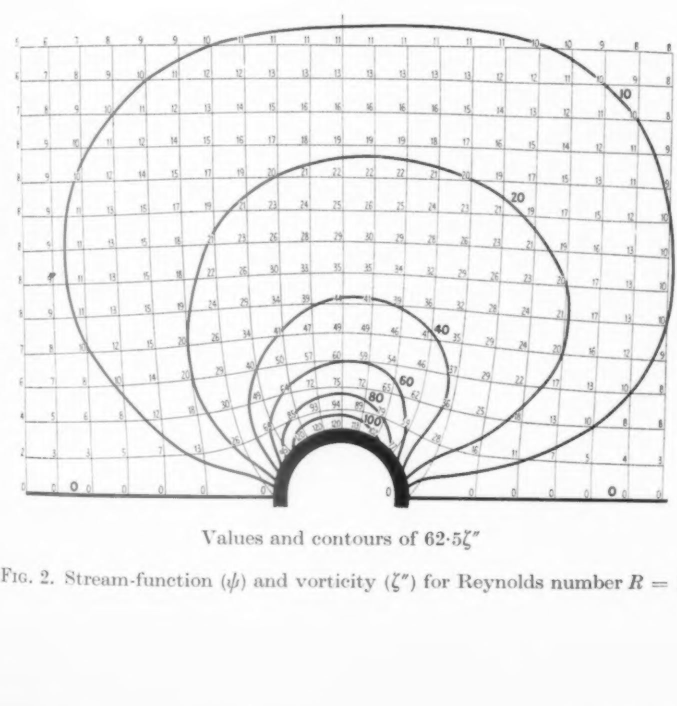


Values and contours of $100\psi''$



Values and contours of $62{\cdot}5\zeta''$

FIG. 2. Stream-function (ψ) and vorticity (ζ'') for Reynolds number $R = 1$.

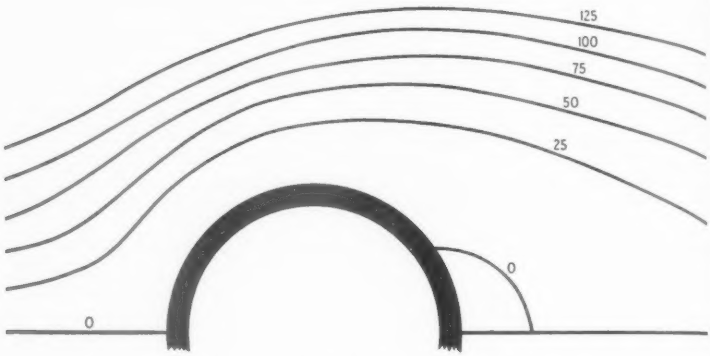


FIG. 3 (*a*). Stream-function ($100\psi''$) for $R = 10$.

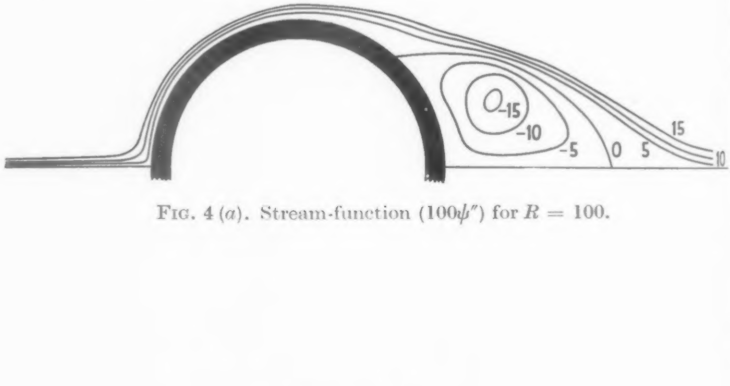


FIG. 4 (*a*). Stream-function ($100\psi''$) for $R = 100$.

FIG. 5 (*a*). Stream-function ($100\psi''$) for $R = 1000$.

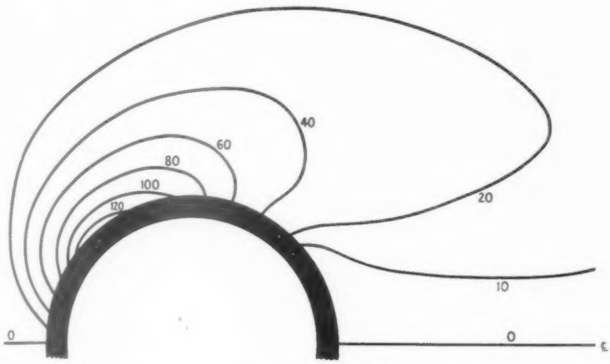


FIG. 3 (*b*). Vorticity ($62{\cdot}5\zeta''$) for $R = 10$.

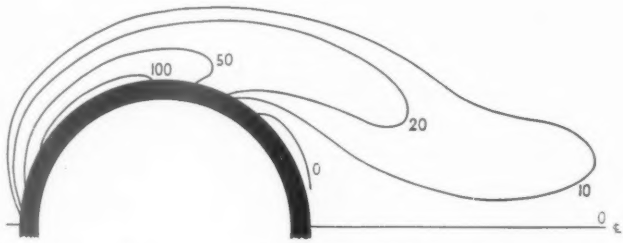


FIG. 4 (*b*). Vorticity ($62{\cdot}5\zeta''$) for $R = 100$.

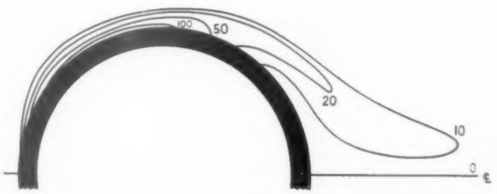


FIG. 5 (*b*). Vorticity ($62{\cdot}5\zeta''$) for $R = 1000$.

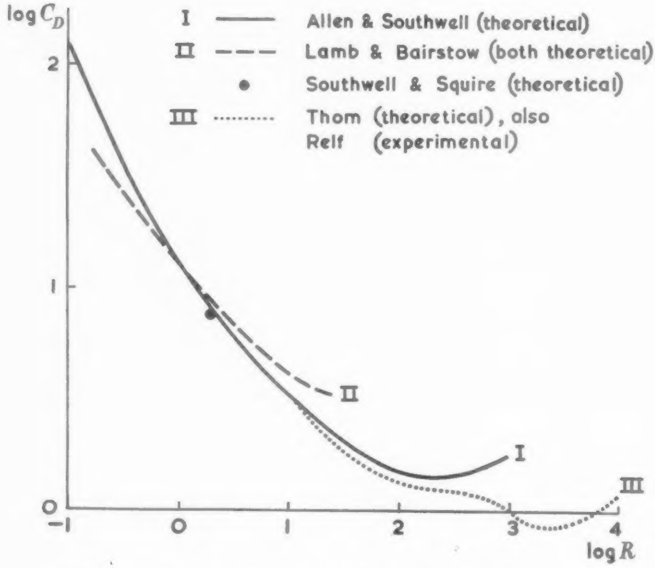


FIG. 6. Drag-coefficient (C_D) as a multiple of $\frac{1}{2}\rho U^2 L$.

FIG. 7. Pressure (p') as a multiple of $\frac{1}{2}\rho U^2$.

6. In this connexion a word should be said about *stability*. It is the essence of the relaxational technique that it starts from solutions which, being inexact, would require body-forces for their maintenance, and thereafter systematically corrects the flow-pattern until the body-forces have been 'liquidated' (rendered negligible). Such treatment, though it does not contemplate instability, will normally detect any tendency of a solution to diverge, and in this way will indicate the occurrence of instability, though it cannot precisely determine the point of transition. It seems worth while to record that in this problem the flow appeared to be stable for $R = 10$, but a distinct impression of instability was gained in the computation for $R = 100$; for (Goldstein *et al.* **3**, p. 419) 'the value of R at which the unsteady régime commences . . . is probably about 50'.†

The instability entailed no computational difficulty: that is to say, steady régimes could be computed for $R = 100$ and for $R = 1000$ which in experiment would not be realizable because the smallest disturbance would upset them.

7. The governing equations. For two-dimensional flow the governing equations are (4)–(9), and the conditions (10) have to be satisfied at the boundary of the inserted cylinder. The relation (4) permits expression of u and v in terms of a stream-function ψ by

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x, \tag{12}$$

and then, according to (9), $\quad \zeta = -\nabla^2\psi \quad (13)$

where ∇^2 has the significance stated in (6). At the boundary of the inserted cylinder ψ must satisfy conditions derived from (10)—viz.

$$\partial\psi/\partial x = 0, \qquad \partial\psi/\partial y = 0. \tag{14}$$

As 'conditions at infinity' we shall assume that

$$\psi \to Uy, \quad \text{so that} \quad \zeta \to 0. \tag{15}$$

8. Reduction of the equations to 'non-dimensional' form. Using L to denote some representative dimension (e.g. the diameter) of the cylinder, and U to denote the velocity at infinity, we now write

$$x = Lx', \quad y = Ly', \quad u = Uu', \quad v = Uv', \quad \psi = UL\psi'. \tag{16}$$

Then, according to (13),

$$\zeta = \frac{U}{L}\zeta', \quad \text{where} \quad -\zeta' = \nabla'^2\psi', \tag{17}$$

† Kawaguti (**2**) found no evidence of instability in his calculations for $R = 40$.

and according to (8)

$$R\left(u'\frac{\partial\zeta'}{\partial x'}+v'\frac{\partial\zeta'}{\partial y'}\right) = \nabla'^2\zeta', \tag{18}$$

where R (the Reynolds number of the motion) $= UL/\nu$ (11) *bis*

and $$\nabla'^2 \equiv \partial^2/\partial x'^2+\partial^2/\partial y'^2. \tag{19}$$

Also, by (12) and (16),

$$u' = \partial\psi'/\partial y', \qquad -v' = \partial\psi'/\partial x',$$

so the boundary conditions (14) and (15) become

$$\left.\begin{array}{l} u' = 0,\ v' = 0, \text{ at the surface of the inserted cylinder,} \\ u' = 1,\ v' = 0, \text{ far away from the inserted cylinder.} \end{array}\right\} \tag{20}$$

9. First change of independent variables (to avoid 'irregular stars'). We can avoid 'irregular stars' in the relaxational computations by changing the variables from x' and y' to α and β, the velocity-potential and stream-function for irrotational flow past the cylinder under discussion.† Far up-stream

$$\alpha \to x'+\text{constant}, \quad \beta \to y', \tag{21}$$

and on the surface of the cylinder (and in the plane of symmetry)

$$\beta = 0. \tag{22}$$

Here α and β are conjugate plane-potential functions such that $(\alpha+i\beta)$ is a function of $(x'+iy')$. Accordingly

$$\frac{\partial\alpha}{\partial x'} = \frac{\partial\beta}{\partial y'}, \qquad -\frac{\partial\alpha}{\partial y'} = \frac{\partial\beta}{\partial x'}, \tag{23}$$

and writing $$h^2 \equiv \left(\frac{\partial\alpha}{\partial x'}\right)^2+\left(\frac{\partial\alpha}{\partial y'}\right)^2 = \left(\frac{\partial\beta}{\partial x'}\right)^2+\left(\frac{\partial\beta}{\partial y'}\right)^2 \tag{24}$$

we have $$\frac{\partial}{\partial x'} = \frac{\partial\alpha}{\partial x'}\frac{\partial}{\partial\alpha}+\frac{\partial\beta}{\partial x'}\frac{\partial}{\partial\beta}, \qquad \frac{\partial}{\partial y'} = \frac{\partial\alpha}{\partial y'}\frac{\partial}{\partial\alpha}+\frac{\partial\beta}{\partial y'}\frac{\partial}{\partial\beta},$$

therefore $$\left.\begin{array}{l} \nabla'^2 \equiv h^2[\partial^2/\partial\alpha^2+\partial^2/\partial\beta^2] = h^2\nabla^2_{\alpha\beta} \text{ (say)}, \\ \text{and} \quad u'\dfrac{\partial}{\partial x'}+v'\dfrac{\partial}{\partial y'} \equiv h^2\left[\dfrac{\partial\psi'}{\partial\beta}\dfrac{\partial}{\partial\alpha}-\dfrac{\partial\psi'}{\partial\alpha}\dfrac{\partial}{\partial\beta}\right]. \end{array}\right\} \tag{25}$$

Consequently (17) transforms to

$$-\zeta' = h^2\nabla^2_{\alpha\beta}\psi', \tag{26}$$

and (18) transforms to

$$R\left(\frac{\partial\psi'}{\partial\beta}\frac{\partial\zeta'}{\partial\alpha}-\frac{\partial\psi'}{\partial\alpha}\frac{\partial\zeta'}{\partial\beta}\right) = \nabla^2_{\alpha\beta}\zeta'. \tag{27}$$

† This device was employed by Thom (**1**). Kawaguti (**2**) employs an iterative process generally similar to Thom's, but a different transformation (into a *finite* rectangle).

The transformed boundary conditions are

$$\left.\begin{aligned} &\psi' = \frac{\partial\psi'}{\partial\beta} = 0 \text{ on that part of the } \alpha\text{-axis } (\beta = 0) \text{ which} \\ &\qquad\qquad \text{corresponds with the inserted cylinder,} \\ &\psi' = \zeta' = 0 \text{ (by symmetry) on all other parts of the } \alpha\text{-axis,} \\ &\frac{\partial\psi'}{\partial\alpha} \to 0, \quad \frac{\partial\psi'}{\partial\beta} \to 1, \quad \zeta' \to 0 \quad \text{as } (\alpha^2+\beta^2) \to \infty. \end{aligned}\right\} \tag{28}$$

10. Second change of variables (to facilitate approximate treatment). The product terms in (26) and (27) imply that the flow-pattern alters with the Reynolds number. But it is evident, even when we envisage 'break-away', that the α-gradients of ψ' and ζ' will not be large, whereas their β-gradients may attain high values; and on that account a further transformation is advantageous.

If β, ψ', ζ' are replaced by β'', ψ'', ζ'', where

$$\beta = R^{-\frac{1}{2}}\beta'', \qquad \psi' = R^{-\frac{1}{2}}\psi'', \qquad \zeta' = R^{\frac{1}{2}}\zeta'', \tag{29}$$

(26) is transformed to

$$-\zeta'' = h^2\nabla''^2\psi'', \tag{30}$$

and (27) to

$$\nabla''^2\zeta'' = \frac{\partial\zeta''}{\partial\alpha}\frac{\partial\psi''}{\partial\beta''} - \frac{\partial\zeta''}{\partial\beta''}\frac{\partial\psi''}{\partial\alpha}, \tag{31}$$

where

$$\nabla''^2 \equiv \frac{1}{R}\nabla^2_{\alpha\beta} \equiv \frac{1}{R}\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta''^2}. \tag{32}$$

Also from (27) and (28) it follows that

$$\left.\begin{aligned} &\psi'' = \frac{\partial\psi''}{\partial\beta''} = 0 \text{ on that part of the } \alpha\text{-axis } (\beta'' = 0) \text{ which} \\ &\qquad\qquad \text{corresponds with the inserted cylinder,} \\ &\psi'' = \zeta'' = 0 \text{ on all other parts of the } \alpha\text{-axis,} \\ &\frac{\partial\psi''}{\partial\alpha} \to 0, \quad \frac{\partial\psi''}{\partial\beta''} \to 1, \quad \zeta'' \to 0, \quad \text{as } (R\alpha^2+\beta''^2) \to \infty. \end{aligned}\right\} \tag{33}$$

Thus R now disappears from the equations, excepting as it enters into ∇''^2; and on the assumption that the derivatives of ψ'' and ζ'' with respect to α and to β'' are comparable, when R is large a close approximation to (32) is

$$\nabla''^2 \doteqdot \partial^2/\partial\beta''^2, \tag{34}$$

which makes the distribution both of ψ'' and of ζ'' independent of R, and thereby greatly reduces the labour of a relaxational treatment. This aims at systematic elimination of residuals which express the errors of a trial solution; and now the errors entailed when the solution for $R = 10$ (say)

is taken as a trial solution for $R = 100$ are small enough to require but little adjustment. Then the solution for $R = 100$ may be utilized, similarly, as a trial solution for $R = 1000$; and so on. (R may be either increased or decreased in successive solutions. Our sequence was $R = 1000$, 100, 0, 1, 10.)

11. Introduction of the relaxation technique. We now describe the application to our present problem of the relaxational techniques which have been explained in earlier papers. They are not required for the transformation of section 9 in relation to a *circular* cylinder, since the known solution can be utilized. It is

$$\left.\begin{aligned} \alpha &= x'\left\{1+\frac{1}{4(x'^2+y'^2)}\right\} \\ \beta &= y'\left\{1-\frac{1}{4(x'^2+y'^2)}\right\} \\ h^2 &= 1+\frac{1-8(x'^2-y'^2)}{16(x'^2+y'^2)^2} \end{aligned}\right\}, \tag{35}$$

when Ox, Oy pass through the axis and when L—the representative dimension—is the diameter of the cylinder. With a use of these expressions the transformation is easy (Fig. 8).

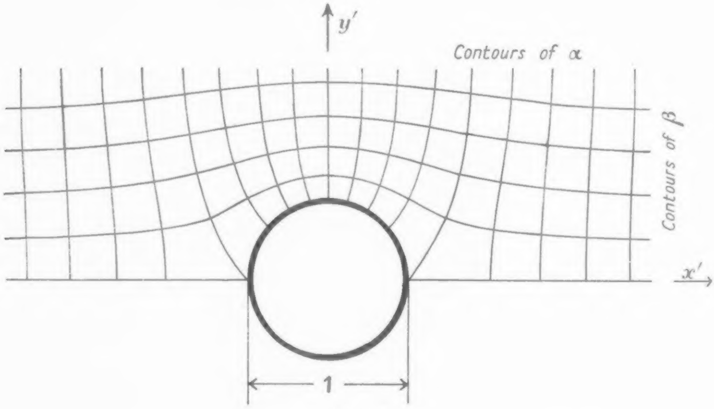


FIG. 8.

12. *Hereafter we shall give to β, ψ, ζ the meanings which in section* 10 *were attached to β'', ψ'', ζ''.* On that understanding (30) will now be written as

$$\frac{\partial^2\psi}{\partial\beta^2}+\frac{1}{R}\frac{\partial^2\psi}{\partial\alpha^2} = -\frac{\zeta}{h^2}, \tag{36}$$

and (31), after some rearrangement, as

$$\frac{\partial^2\zeta}{\partial\beta^2}+\left(\frac{\partial\psi}{\partial\alpha}\right)\frac{\partial\zeta}{\partial\beta}+\frac{1}{R}\left\{\frac{\partial^2\zeta}{\partial\alpha^2}-R\frac{\partial\psi}{\partial\beta}\frac{\partial\zeta}{\partial\alpha}\right\}=0. \tag{37}$$

It is not necessary to rewrite the boundary conditions (33).

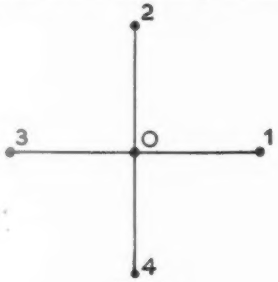


FIG. 9.

For a relaxational treatment, (36) and (37) must be replaced by approximations in finite differences. Since (36) is linear, it presents no difficulty: in the notation of 'residuals' (4) its approximation is

$$(F_\psi)_0 \equiv \psi_2+\psi_4-2\psi_0+\frac{1}{R}(\psi_1+\psi_3-2\psi_0)+a^2(\zeta/h^2)_0 = 0, \tag{38}$$

the suffixes 0, 1, 2, 3, and 4 denoting points so numbered in Fig. 9;

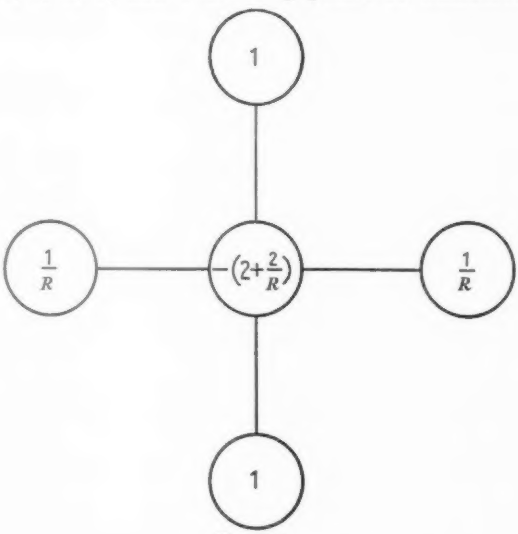


FIG. 10.

and from (38) it is easy to deduce the standard 'pattern' shown in Fig. 10, for the effect on the F_ψ's of an increment $\Delta\psi = 1$ at 0, Fig. 9.

13.† Equation (37) presents a harder problem, and being non-linear yields a 'pattern' which alters as the work proceeds. But it has been shown in earlier papers that inaccurate patterns can be used to liquidate residuals, provided that an exact account of these is kept; and here a simple pattern represents sufficiently a relation of rather complicated form.

We contemplate a 'two-diagram technique' in which ψ and ζ are modified alternately. At the end of a stage of ζ-relaxation the F_ψ's will have altered in accordance with the last term of (38), and the ψ's must be altered so as (temporarily) to liquidate them: then, with the altered values given to ψ, the ζ's must be modified in accordance with a finite-difference approximation to (37). This we proceed to derive.

14. The two quantities which together make up (37)—namely,

$$\left.\begin{aligned} &(a) \qquad \frac{\partial^2\zeta}{\partial\beta^2}+\frac{\partial\psi}{\partial\alpha}\frac{\partial\zeta}{\partial\beta} \\ \text{and} \qquad &(b) \qquad \frac{\partial^2\zeta}{\partial\alpha^2}-R\frac{\partial\psi}{\partial\beta}\frac{\partial\zeta}{\partial\alpha} \end{aligned}\right\} \tag{39}$$

—can be treated similarly in a way we now explain in relation to (*a*). Writing

$$\kappa \equiv \frac{\partial\psi}{\partial\alpha}, \qquad A \equiv \frac{\partial^2\zeta}{\partial\beta^2}+\kappa\frac{\partial\zeta}{\partial\beta}, \tag{40}$$

we have as the solution of the second of (40) *when κ and A are invariant*

$$\kappa\zeta = A\beta+P+Qe^{-\kappa\beta},$$

P and Q being constants of integration. Then, 2, 0 and 4 denoting adjacent nodes on a β-line of the α–β net (so that

$$\beta_2 = \beta_0+a, \qquad \beta_4 = \beta_0-a$$

when a, as is usual, denotes the mesh-length), we have

$$\kappa(\zeta_2-\zeta_0) = Aa+Qe^{-\kappa\beta_0}(e^{-\kappa a}-1),$$

$$\kappa(\zeta_4-\zeta_0) = -Aa+Qe^{-\kappa\beta_0}(e^{\kappa a}-1),$$

therefore
$$\kappa\{e^{\kappa a}(\zeta_2-\zeta_0)+\zeta_4-\zeta_0\} = Aa(e^{\kappa a}-1).$$

We thus have an expression for A—i.e. for (*a*) of (39); and a like expression may be formed in the same way for (*b*). The values of κ ($\equiv \partial\psi/\partial\alpha$) and of λ ($\equiv -R\,\partial\psi/\partial\beta$), the corresponding quantity in the expression for (*b*), will of course not be known until the solution has been completed; but they may be treated as known when ψ has been *temporarily* determined.

† It should be recorded that the technique described in Sections 13–16 is entirely due to Allen (R.V.S.).

and values appropriate to the point 0 may be computed from finite-difference expressions of normal form: viz. from

$$2\kappa a = \psi_1 - \psi_3, \qquad 2\lambda a = -R(\psi_2 - \psi_4).$$

We then have as an expression for the typical ζ-residual, defined as the finite-difference approximation to $a^2 \times$[left-hand side of (37)]:

$$(F_\zeta)_0 \equiv \kappa_0 a\{e^{\kappa_0 a}(\zeta_2 - \zeta_0) + \zeta_4 - \zeta_0\}/(e^{\kappa_0 a} - 1) + \\ + \lambda_0 a\{e^{\lambda_0 a}(\zeta_1 - \zeta_0) + \zeta_3 - \zeta_0\}/R(e^{\lambda_0 a} - 1). \quad (41)$$

(It is the latent exponentials in the solution, revealed in (41), which necessitate this special treatment. Except within a very narrow range, an exponential is not closely represented by a polynomial; and on that account the customary expression in finite differences for the second term in (39), (a) or (b), has insufficient accuracy when κ_0 and/or λ_0 is not small.)

15. Regard must be paid, in liquidation, to the boundary conditions (33), section 10. Rewritten in accordance with section 12, and with

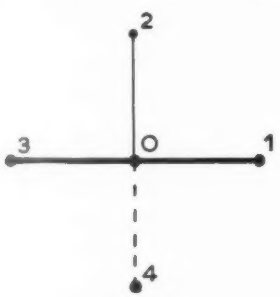


FIG. 11.

derivatives replaced by their approximations in finite differences, these become

at all nodes on the α-axis except the 'slit' (i.e. that part which corresponds with the inserted cylinder):

$$\psi = \zeta = 0;$$

at nodes far away from the 'slit' (where $\alpha^2 + \beta^2 \to \infty$):

$$\psi \to \beta, \qquad \zeta \to 0;$$

at nodes on the 'slit' (typified by 0, Fig. 11, in which the node 4 is 'fictitious'):

$$\psi_0 = 0, \qquad \psi_2 = \psi_4. \quad (42)$$

Only the last of (42) needs special notice. Combined with (38) of section 12 (which here reduces to

$$\psi_2+\psi_4+\frac{a^2}{h_0^2}\zeta_0 = 0$$

because $\psi_3 = \psi_0 = \psi_1 = 0$), it becomes

$$\zeta_0 = -2h_0^2\psi_2/a^2, \tag{43}$$

so permits, at the start of each stage of 'ζ-relaxation', a specification of ζ at all nodes on the α-axis.

16. Except in its use of 'patterns' which alter as the work proceeds and which call for a use of exponential tables, the relaxational procedure follows normal lines; 'residuals' of the types $\mathbf{F}_\psi$ and $\mathbf{F}_\zeta$ being liquidated *alternately, in stages.* Small changes made in ζ-values were found to alter ψ-values largely, and on that account the F_ζ's were liquidated more or less completely; but the stages of 'ψ-relaxation' were made short because the F_ψ's became smaller almost automatically by reason of the changes made in the ζ's.

17. When acceptable distributions have been found for ψ and ζ as functions of α and β—that is, in the notation of section 10, for ψ'' and ζ'' as functions of α and β''—the transformations (29) will yield ψ' and ζ' in terms of α and β, which have the expressions (35) in terms of x' and y'. Contours of ψ'' and ζ'' can then be plotted on the rectangular (x', y') net. Figs. 1–5 were thus derived.

For Figs. 6 and 7, expressions for the boundary tractions were required. It can be shown (cf., e.g., (**5**), sections 325–6) that the normal pressure on the circular cylinder is

$$-p_{nn}\text{ (say)} = p-2\nu\rho\cos 2\theta\frac{\partial^2\psi}{\partial x\partial y}+\nu\rho\sin 2\theta\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right]\psi$$

and the tangential traction (upstream) on the cylinder is (44)

$$p_{ns}\text{ (say)} = \nu\rho\cos 2\theta\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right]\psi+2\nu\rho\sin 2\theta\frac{\partial^2\psi}{\partial x\partial y}$$

at a point whose angular distance from the upstream stagnation-point is θ. At points on the boundary $\left(\text{where } \psi = \frac{\partial\psi}{\partial\alpha} = \frac{\partial^2\psi}{\partial\alpha^2} = 0, \text{ so } \zeta = h^2\frac{\partial^2\psi}{\partial\beta^2}\right)$

$$\frac{\partial^2\psi}{\partial x\partial y} \equiv \frac{\partial^2\alpha}{\partial x^2}\frac{\partial\psi}{\partial\beta}+\frac{\zeta}{h^2}\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y}+\left\{\left(\frac{\partial\alpha}{\partial x}\right)^2-\left(\frac{\partial\alpha}{\partial y}\right)^2\right\}\frac{\partial^2\psi}{\partial\alpha\partial\beta},$$

$$\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2} \equiv \zeta\left\{1-\frac{2}{h^2}\left(\frac{\partial\alpha}{\partial x}\right)^2\right\}+2\frac{\partial^2\alpha}{\partial x\partial y}\frac{\partial\psi}{\partial\beta}+4\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y}\frac{\partial^2\psi}{\partial\alpha\partial\beta},$$

and p, the 'mean normal pressure' (section 1), is related with p_s, the 'static pressure at infinity', by

$$p_s - p = \tfrac{1}{2}\rho h^2\left(\frac{\partial\psi}{\partial\beta}\right)^2 + \mu \int\limits_{-\infty} \frac{\partial\zeta}{\partial\beta}\,d\alpha. \tag{45}$$

Consequently

$$\left.\begin{aligned} -p_{nn} &= p_s - \rho U^2 \int\limits_{-\infty} \frac{\partial\zeta''}{\partial\beta''}\,d\alpha \\ p_{ns} &= \rho U^2\zeta''/R^{\frac{1}{2}} \end{aligned}\right\}, \tag{46}$$

at a point on the boundary (where, by (35), $\alpha = 2x' = -\cos\theta$, $\beta = 0$). From (45) and (46) the 'drag' (Fig. 6) and pressure-distribution (Fig. 7) were computed: Fig. 6 by integration of

$$-(p_{ns}\sin\theta + p_{nn}\cos\theta).$$

18. This investigation, started in 1944, was frequently interrupted on account of pressure of more immediately urgent problems. An interim account of it was presented at the International Congress of Applied Mechanics (London) in 1948.

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REFERENCES

1. A. Thom, *Proc. Roy. Soc.* A, **141** (1933), 651–69.
2. M. Kawaguti, *J. Phys. Soc. Japan*, **8** (1953), 747–57.
3. S. Goldstein *et al.*, *Modern Developments in Fluid Dynamics* (1st ed.) (Oxford, 1948).
4. R. V. Southwell, *Relaxation Methods in Theoretical Physics* (1st ed.) (Oxford, 1946).
5. H. Lamb, *Hydrodynamics* (6th ed.) (Cambridge, 1932).

ROTATORY AND LONGITUDINAL OSCILLATIONS OF AXI-SYMMETRIC BODIES IN A VISCOUS FLUID

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SUMMARY

The Stokes stream function is used to obtain the perturbations arising from the slow rotatory and longitudinal oscillations of axi-symmetrical bodies in an infinite mass of viscous fluid which is at rest at infinity. The bodies considered in this paper are a sphere, an infinite circular cylinder, a prolate spheroid, an oblate spheroid, and a circular disk. The case of the circular disk is deduced as a limiting case of the oblate spheroid. The paper has been divided into two parts. Part I deals with the rotatory oscillations and Part II with the longitudinal ones.

1. Introduction

THE perturbations arising from the longitudinal vibrations of a sphere along a diameter in an infinite mass of viscous fluid at rest have been considered by the author (**1**) by applying the idea of periodic singular points and the results are found to agree with those found by Lamb (**2**). The rotatory oscillations of a sphere about a diameter, in an infinite mass of viscous fluid at rest, have been discussed by Lamb (**2**). The object of this paper is to utilize the Stokes stream function to obtain the motion of the fluid due to the rotatory and longitudinal oscillations of certain axi-symmetric bodies in a viscous fluid. The slow motion, in which the inertia terms in the equations of motion are neglected, is considered in the case when the bodies are in the form of a sphere, an infinite circular cylinder, a prolate spheroid, an oblate spheroid, and a circular disk. The rotatory oscillations are about a diameter in the case of a sphere, about its axis in the case of an infinite circular cylinder, about their axes of symmetry in the cases of spheroids, and about the normal axis through its centre in the case of a circular disk. The case of the circular disk has been discussed as a limiting case of the oblate spheroid. The longitudinal oscillations are along the same axes about which the rotatory oscillations of respective bodies have been considered. Some of the results obtained agree with results already known, while others appear to be new. For spheroids it is found that we have to use spheroidal wave functions discussed by Stratton, Morse, Chu, and Hutner (**3**). The notation given by them will be used throughout this paper.

2. Equations of motion

Let α, β, γ be the general orthogonal coordinates and let the elements of length at the point (α, β, γ) in the directions of α, β, γ increasing, respectively, be $e_1 d\alpha$, $e_2 d\beta$, and $e_3 d\gamma$, such that

$$ds^2 = e_1^2(d\alpha)^2 + e_2^2(d\beta)^2 + e_3^2(d\gamma)^2.$$

Let u, v, w be respectively the components of the velocity in the directions of α, β, γ increasing. For motion symmetrical about an axis (**4**) we take α and β to be the general orthogonal coordinates in a meridian plane and γ the azimuthal angle ϕ, so that e_3 will be the distance from the axis of revolution. All quantities are supposed to be independent of ϕ. The equation of continuity then takes the form

$$\frac{\partial}{\partial \alpha}(e_2 e_3 u) + \frac{\partial}{\partial \beta}(e_3 e_1 v) = 0, \tag{1}$$

so that there is a stream-function ψ such that

$$e_3 u = \frac{1}{e_2}\frac{\partial \psi}{\partial \beta}, \qquad e_3 v = -\frac{1}{e_1}\frac{\partial \psi}{\partial \alpha}. \tag{2}$$

This is true whether the velocity w round the axis is zero or not so long as it is independent of ϕ. In the case when there is an azimuthal velocity w, we put

$$e_3 w = \Omega;$$

then ξ, η, ζ, the components of vorticity, are given by

$$\xi = \frac{1}{e_2 e_3}\frac{\partial \Omega}{\partial \beta}, \qquad \eta = -\frac{1}{e_3 e_1}\frac{\partial \Omega}{\partial \alpha},$$

and
$$\zeta = -\frac{1}{e_1 e_2}\left[\frac{\partial}{\partial \alpha}\left(\frac{e_2}{e_3 e_1}\frac{\partial \psi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{e_1}{e_3 e_2}\frac{\partial \psi}{\partial \beta}\right)\right] = -\frac{1}{e_3}D^2\psi,$$

where
$$D^2 = \frac{e_3}{e_1 e_2}\left[\frac{\partial}{\partial \alpha}\left(\frac{e_2}{e_3 e_1}\frac{\partial}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{e_1}{e_3 e_2}\frac{\partial}{\partial \beta}\right)\right].$$

The equations of motion give

$$\frac{\partial}{\partial t}(D^2\psi) + \frac{2\Omega}{e_1 e_2 e_3^2}\frac{\partial(\Omega, e_3)}{\partial(\alpha, \beta)} - \frac{1}{e_1 e_2 e_3}\frac{\partial(\psi, D^2\psi)}{\partial(\alpha, \beta)} + \frac{2D^2\psi}{e_1 e_2 e_3^2}\frac{\partial(\psi, e_3)}{\partial(\alpha, \beta)} = \nu D^4\psi,$$

which, if we neglect the second-order terms, becomes

$$\frac{\partial}{\partial t}(D^2\psi) = \nu D^4\psi, \tag{3}$$

which is the equation for ψ; likewise, the equation for Ω is

$$\frac{\partial \Omega}{\partial t} - \frac{1}{e_1 e_2 e_3}\frac{\partial(\psi, \Omega)}{\partial(\alpha, \beta)} = \nu D^2\Omega. \tag{4}$$

(*a*) *Rotatory oscillations*. In this case

$$u = 0, \qquad v = 0, \qquad e_3 w = \Omega.$$

Thus $\psi = 0$ and the equation (3) is automatically satisfied. The motion is given by the equation (4), which reduces to

$$\frac{\partial \Omega}{\partial t} = \nu D^2 \Omega. \tag{5}$$

(*b*) *Longitudinal oscillations*. In this case

$$e_3 u = \frac{1}{e_2}\frac{\partial \psi}{\partial \beta}, \qquad e_3 v = -\frac{1}{e_1}\frac{\partial \psi}{\partial \alpha}, \qquad w = 0.$$

Hence $\Omega = 0$, the equation (4) is satisfied, and the motion is given by the equation (3), which now may be written as

$$D^2\left(D^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\psi = 0. \tag{6}$$

If ψ_1 and ψ_2 be two functions satisfying the equations

$$D^2 \psi_1 = 0, \tag{7}$$

$$\left(D^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\psi_2 = 0, \tag{8}$$

a solution of the equation (6) may be written as

$$\psi = \psi_1 + \psi_2. \tag{9}$$

Thus we have to solve the equations (7) and (8). The superposition of their solutions will give the required stream function.

To satisfy the conditions at the surface of the body in this case we take the origin (**2**) at the mean position of the centre of the body.

Part I: Rotatory Oscillations

I. *Sphere oscillating about a diameter*

3. Taking spherical polar coordinates R, θ, ϕ for α, β, γ respectively we get

$$e_1 = 1, \qquad e_2 = R, \qquad e_3 = R\sin\theta,$$

and

$$u = 0, \qquad v = 0, \qquad w = \Omega/R\sin\theta.$$

The equation of motion for harmonic oscillations of period $2\pi/\sigma$ is

$$\frac{\partial^2 \Omega_1}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \Omega_1}{\partial \theta}\right) + h^2 \Omega_1 = 0, \tag{10}$$

where

$$\Omega = \Omega_1 e^{i\sigma t}, \qquad h^2 = -i\sigma/\nu.$$

The solution satisfying the boundary condition at infinity is

$$\Omega_1 = A e^{-ihR}\left(1 + \frac{1}{ihR}\right)\sin^2\theta, \tag{11}$$

and the angular velocity ω is given as

$$\omega = \frac{Ae^{-ihR}}{R^2}\left(1+\frac{1}{ihR}\right)e^{i\sigma t}. \tag{12}$$

Applying the boundary condition at $R = a$, viz.

$$\omega = \omega_0 e^{i\sigma t}, \tag{13}$$

we find

$$A = \frac{ih\omega_0 e^{iha}a^3}{(1+iha)}. \tag{14}$$

4. The couple required to maintain the motion

The tangential stress on the surface $R =$ constant and in the direction of ϕ increasing is

$$T = \frac{\mu\omega_0(3+3iha-h^2a^2)}{(1+iha)}\sin\theta\, e^{i\sigma t}. \tag{15}$$

The couple G is thus given by

$$G = -\left\{\tfrac{8}{3}\pi\mu a^3\omega_0\frac{3+3iha-h^2a^2}{1+iha}\right\}e^{i\sigma t}, \tag{16}$$

which agrees with the result obtained otherwise by Lamb (**2**).

II. *Circular cylinder oscillating about its axis*

5. For cylindrical polar coordinates we take α as z and β as r. Then the equation of motion for harmonic oscillations reduces to

$$\frac{d^2\Omega_1}{dr^2}-\frac{1}{r}\frac{d\Omega_1}{dr}-ik^2\Omega_1 = 0, \tag{17}$$

where $\qquad k^2 = \sigma/\nu \quad$ and $\quad \Omega = \Omega_1 e^{i\sigma t}$.

Its appropriate solution is given in terms of modified Bessel function as

$$\Omega_1 = ArK_1(\sqrt{i}\,kr), \tag{18}$$

and the angular velocity ω is given by

$$\omega = \frac{AK_1(\sqrt{i}\,kr)}{r}e^{i\sigma t}. \tag{19}$$

The boundary condition at $r = a$, viz. $\omega = \omega_0 e^{i\sigma t}$, gives

$$A = \frac{\omega_0 a}{K_1(\sqrt{i}\,ka)}. \tag{20}$$

6. The couple required to maintain the motion

The tangential stress T on the surface $r = a$ and in the direction of ϕ increasing is given by

$$T = \frac{-\mu\omega_0\sqrt{i}\,kaK_2(\sqrt{i}\,ka)}{K_1(\sqrt{i}\,ka)}e^{i\sigma t}. \tag{21}$$

So the couple per unit length of the cylinder is

$$G = -2\pi\mu\omega_0 a^2 \frac{\sqrt{i}\,kaK_2(\sqrt{i}\,ka)}{K_1(\sqrt{i}\,ka)} e^{i\sigma t}. \tag{22}$$

III. *Prolate spheroid oscillating about its axis of revolution*

7. In this case we use the prolate spheroidal coordinates defined by

$$z+ir = c\cosh(\alpha+i\beta),$$

$$e_1 = e_2 = c\sqrt{(\cosh^2\alpha-\cos^2\beta)}, \qquad e_3 = c\sinh\alpha\sin\beta.$$

The equation of motion is $\qquad \dfrac{\partial\Omega}{\partial t} = \nu D^2\Omega,$

where

$$D^2 = \left\{\frac{1}{c^2(\cosh^2\alpha-\cos^2\beta)}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\coth\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}\right)\right\}.$$

Writing $\Omega_1 e^{i\sigma t} = \Omega$ in the equation of motion, we get

$$D^2\Omega_1 = \frac{i\sigma}{\nu}\Omega_1 = ik^2\Omega_1, \tag{23}$$

where $k^2 = \sigma/\nu$, or

$$\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\coth\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}-ik^2c^2(\cosh^2\alpha-\cos^2\beta)\right]\Omega_1 = 0. \tag{24}$$

Putting

$$\Omega_1 = \sinh\alpha\sin\beta\, S(\beta)R(\alpha),$$

$$\cosh\alpha = \xi, \qquad \cos\beta = \eta,$$

equation (24) separates into the two following equations:

$$\frac{d}{d\eta}\left\{(\eta^2-1)\frac{dS}{d\eta}\right\}+\left\{\lambda-ik^2c^2\eta^2-\frac{1}{\eta^2-1}\right\}S = 0, \tag{25}$$

$$\frac{d}{d\xi}\left\{(\xi^2-1)\frac{dR}{d\xi}\right\}+\left\{\lambda-ik^2c^2\xi^2-\frac{1}{\xi^2-1}\right\}R = 0, \tag{26}$$

where λ is the constant of separation. Equations (25) and (26) are identical with the equations obtained by Stratton and others (**3**) in the discussion of the prolate spheroidal wave functions. Such equations contain a third parameter m and have solutions for the suitably related values of the parameter in series of associated Legendre functions or Bessel functions of half order. In our case m is unity. Thus, for various characteristic values of λ, there are 'angular solutions' $S^1_{1l}(\sqrt{i}\,kc, \eta)$ of (25) which are finite throughout the range $-1 \leqslant \eta \leqslant 1$ and are expressed as infinite series of associated Legendre functions in the form

$$S^1_{1l}(\sqrt{i}\,kc, \eta) = \sum_{n=0,1}{}' d^l_n P^1_{n+1}(\eta), \tag{27}$$

where the prime indicates summation over even or odd values of n according as n is even or odd, and

$$\sum_{n=0}^{\infty}{}' i^{n-l}\, d_n^l \frac{\{\frac{1}{2}(n+1)\}!}{(\frac{1}{2}n)!} = \frac{\{\frac{1}{2}(l+1)\}!}{(\frac{1}{2}l)!}, \quad l \text{ even},$$

$$\sum_{n=1}^{\infty}{}' i^{n-l}\, d_n^l \frac{\{\frac{1}{2}(n+2)\}!}{\{\frac{1}{2}(n-1)\}!} = \frac{\{\frac{1}{2}(l+2)\}!}{\{\frac{1}{2}(l-1)\}!}, \quad l \text{ odd}.$$

Since Ω is to be expressed ultimately as the sum of series of products $S(\beta)R(\alpha)$, the boundary condition $\Omega \to 0$ as $\xi \to \infty$ determines at once the appropriate 'radial solution' of (26). The function which satisfies this condition is $R_{1l}^3(\sqrt{i}\,kc, \xi)$, which is given, for large values of $\sqrt{i}\,kc\xi$, as

$$R_{1l}^{(3)}(\sqrt{i}\,kc, \xi) = \frac{(\xi^2-1)^{\frac{1}{2}}\sqrt{(\pi/2\sqrt{i}\,kc\xi^3)} \sum_{n=0,1}^{\infty}{}' i^{l-n} d_n^l (n+1)(n+2) K_{n+\frac{3}{2}}(\sqrt{i}\,kc\xi)}{\sum_{n=0,1}^{\infty}{}' d_n^l (n+1)(n+2)} \tag{28}$$

$$\to \frac{1}{\sqrt{i}\,kc\xi} e^{i\{\sqrt{i}\,kc\xi - \frac{1}{2}(l+2)\pi\}}. \tag{29}$$

The complete solution of equation (24) is

$$\Omega_1 = c \sinh\alpha \sin\beta \sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha) S_{1l}^1(\beta). \tag{30}$$

Therefore
$$\omega = \sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha) S_{1l}^1(\beta) e^{i\sigma t} \tag{31}$$

and the angular velocity is

$$\omega = \frac{1}{c \sinh\alpha \sin\beta} \sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha) S_{1l}^1(\beta) e^{i\sigma t}. \tag{32}$$

The boundary condition requires

$$\omega = \omega_0 e^{i\sigma t} \quad \text{at } \alpha = \alpha_0.$$

Therefore
$$\omega_0 c \sinh\alpha_0 \sin\beta = \sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha_0) S_{1l}^1(\beta). \tag{33}$$

Using the orthogonal property of $S_{1l}^1(\beta)$, we have

$$\omega_0 c \sinh\alpha_0 \int_0^{\pi} \sin^2\beta\, S_{1l}^1(\beta)\, d\beta = A_l q_l\, R_{1l}^3(\alpha_0), \tag{34}$$

where
$$q_l = 2 \sum_{n=0,1}^{\infty}{}' \frac{(n+1)(n+2)}{(2n+3)} (d_n^l)^2. \tag{35}$$

To integrate $$\int_0^\pi \sin^2\beta \, S_{1l}^1(\beta)\, d\beta = \int_{-1}^{+1} \sqrt{(1-\eta^2)}\, S_{1l}^1(\eta)\, d\eta,$$

where $$\eta = \cos\beta,$$

we notice that $$S_{1l}^1(\eta) = \sum_{n=0,1}^{\infty}{}' \, d_n^l \, P_{n+1}^1(\eta),$$

and assume the validity of the interchange of the operations of integration and summation. We have

$$\int_{-1}^{+1} \sqrt{(1-\eta^2)} P_{n+1}^1(\eta)\, d\eta = -\int_{-1}^{+1} P_1^1(\eta) P_{n+1}^1(\eta)\, d\eta = \begin{cases} -\frac{4}{3} & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}$$

So the equation (34) gives

$$\left.\begin{aligned} A_l &= \frac{-4\omega_0 c \sinh\alpha_0}{3q_l R_{1l}^3(\alpha_0)} \quad && \text{for } l \text{ even} \\ A_l &= 0 && \text{for } l \text{ odd} \end{aligned}\right\}. \tag{36}$$

8. The couple required to maintain the motion

The tangential stress T on the surface $\alpha = \alpha_0$ and in the direction of α increasing is given by

$$\begin{aligned} T &= \mu \left\{ \frac{e_3}{e_1} \left[\frac{\partial}{\partial\alpha} \left(\frac{\omega}{e_3} \right) \right] \right\}_{\alpha=\alpha_0} \\ &= \frac{\mu \sinh\alpha_0}{c(\cosh^2\alpha_0 - \cos^2\beta)^{\frac{1}{2}}} \left[\frac{\partial}{\partial\alpha} \left\{ \frac{\sum_{l=0}^{\infty} A_l R_{1l}^3(\alpha) S_{1l}^1(\beta)}{\sinh\alpha} \right\} \right]_{\alpha=\alpha_0} e^{i\sigma t} \\ &= -\left\{ \frac{\mu\omega_0 \cosh\alpha_0 \sin\beta}{(\cosh^2\alpha_0 - \cos^2\beta)^{\frac{1}{2}}} + \frac{4\mu\omega_0 \sum_{l=0}^{\infty} \{R_{1l}^{3\prime}(\alpha_0)/q_l R_{1l}^3(\alpha_0)\} S_{1l}^1(\beta)}{3(\cosh^2\alpha_0 - \cos^2\beta)^{\frac{1}{2}}} \right\} e^{i\sigma t}. \end{aligned} \tag{37}$$

The required couple G is given as

$$\begin{aligned} G &= -e^{i\sigma t} \Big[2\pi\mu\omega_0 c^3 \sinh^2\alpha_0 \cosh\alpha_0 \int_0^\pi \sin^3\beta\, d\beta + \\ &\qquad + \tfrac{8}{3}\pi\mu\omega_0 c^3 \sinh^2\alpha_0 \sum_{l=0}^{\infty} \{R_{1l}^{3\prime}(\alpha_0)/q_l R_{1l}^3(\alpha_0)\} \int_0^\pi \sin^2\beta\, S_{1l}^1(\beta)\, d\beta \Big] \\ &= \Big[-\cosh\alpha_0 + \tfrac{4}{3} \sum_{l=0}^{\infty} \{R_{1l}^{3\prime}(\alpha_0)/q_l R_{1l}^3(\alpha_0)\} \Big] \tfrac{8}{3}\pi\mu\omega_0 c^3 \sinh^2\alpha_0 \, e^{i\sigma t}, \end{aligned} \tag{38}$$

where a dash denotes differentiation with respect to α.

IV. *Oblate spheroid oscillating about its axis of revolution*

For an oblate spheroid we introduce a system of coordinates defined by

$$z+ir = c\sinh(\alpha+i\beta),$$
$$\phi = \gamma.$$

We then get

$$e_1 = e_2 = c\sqrt{\{\sinh^2\alpha+\cos^2\beta\}}, \qquad e_3 = c\cosh\alpha\sin\beta.$$

The equation of motion $$\frac{\partial\Omega}{\partial t} = \nu D^2\Omega$$

transforms into

$$\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\tanh\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}-ik^2c^2(\sinh^2\alpha+\cos^2\beta)\right]\Omega_1 = 0, \tag{39}$$

where $$\Omega = \Omega_1 e^{i\sigma t}, \qquad k^2 = \sigma/\nu.$$

Putting

$$\Omega_1 = \cosh\alpha\sin\beta\, S(\beta)R(\alpha),$$
$$\sinh\alpha = \xi, \quad \text{and} \quad \cos\beta = \eta,$$

equation (39) separates into the following two equations,

$$\frac{d}{d\eta}\left[(\eta^2-1)\frac{dS}{d\eta}\right]+\left[\lambda+ik^2c^2\eta^2-\frac{1}{\eta^2-1}\right]S = 0, \tag{40}$$

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{dR}{d\xi}\right]+\left[\lambda-ik^2c^2\xi^2+\frac{1}{\xi^2+1}\right]R = 0, \tag{41}$$

where λ is a constant of separation. Equation (40) can be obtained from (25) by changing $\sqrt{i}\,k$ to $i\sqrt{i}\,k$. The equation (41) can be obtained from equation (26) by changing $\sqrt{i}\,k$ to $i\sqrt{i}\,k$ and ξ to $-i\xi$. Therefore the solutions of the equations (40) and (41) can be obtained from those of the equations (25) and (26) by changing $\sqrt{i}\,k$ to $-i\sqrt{i}\,k$ and ξ to $i\xi$. Thus the angular solution is given by

$$S^1_{1l}(-i\sqrt{i}\,k,\eta) = \sum_{n=0,1}^{\infty}{}' f^l_n P^1_{n+1}(\eta), \tag{42}$$

where the coefficients f^l_n are different in value from the coefficients d^l_n of the prolate case, but are obtained in the same way (5). The radial solutions are given by

$$R^3_{1l}(-i\sqrt{i}\,kc, i\xi)$$
$$= \frac{(1+\xi^2)^{\frac{1}{2}}\sqrt{\{\pi/(-2\sqrt{i}\,kc\xi^3)\}}\sum_{n=0,1}^{\infty}{}' i^{l-n}f^l_n(n+1)(n+2)K_{n+\frac{3}{2}}(\sqrt{i}\,kc\xi)}{\sum_{n=0,1}^{\infty}{}' f^l_n(n+1)(n+2)}. \tag{43}$$

The required solution in this case is

$$\Omega = c\cosh\alpha\sin\beta\sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha)S_{1l}^1(\beta)e^{i\sigma t}, \tag{44}$$

$$w = \sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha)S_{1l}^1(\beta)e^{i\sigma t}, \tag{45}$$

$$\omega = \frac{1}{c\cosh\alpha\sin\beta}\sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha)S_{1l}^1(\beta)e^{i\sigma t}. \tag{46}$$

The boundary condition gives

$$\omega_0 c\cosh\alpha_0\sin\beta = \sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha_0)S_{1l}^1(\beta). \tag{47}$$

Employing the orthogonal property of $S_{1l}^1(\beta)$, we get, as in equation (33),

$$\left.\begin{aligned} A_l &= \frac{-4\omega_0 c\cosh\alpha_0}{3q_l\, R_{1l}^3(\alpha_0)} && \text{for } l \text{ even} \\ A_l &= 0 && \text{for } l \text{ odd} \end{aligned}\right\}, \tag{48}$$

where $$q_l = 2\sum_{n=0,1}^{\infty}{}' \frac{(n+1)(n+2)}{(2n+3)}(f_n^l)^2.$$

9. The couple required to maintain the motion

T, the tangential stress on the surface $\alpha = \alpha_0$ and in the direction of ϕ increasing, is

$$\begin{aligned} T &= \mu\frac{e_3}{e_1}\left[\frac{\partial}{\partial\alpha}\left(\frac{w}{e_3}\right)\right]_{\alpha=\alpha_0} \\ &= \frac{\mu\cosh\alpha_0}{c(\sinh^2\alpha_0+\cos^2\beta)^{\frac{1}{2}}}\left[\frac{\partial}{\partial\alpha}\left\{\operatorname{sech}\alpha\sum_{l=0}^{\infty} A_l\, R_{1l}^3(\alpha)S_{1l}^1(\beta)\right\}e^{i\sigma t}\right]_{\alpha=\alpha_0}. \end{aligned} \tag{49}$$

The required couple is given as

$$\begin{aligned} G &= \left[2\pi\mu c^2\cosh^2\alpha_0\sum_{l=0}^{\infty} A_l\, R_{1l}^{3\prime}(\alpha_0)\int_0^{\pi}\sin^2\beta\, S_{1l}^1(\beta)\,d\beta - \right. \\ &\qquad \left. -2\pi\mu c^3\omega_0\cosh^2\alpha_0\sinh\alpha_0\int_0^{\pi}\sin^3\beta\,d\beta\right]e^{i\sigma t} \\ &= \left[\tfrac{32}{9}\pi\mu\omega_0 c^3\cosh^3\alpha_0\sum_{l=0}^{\infty}\frac{R_{1l}^{3\prime}(\alpha_0)}{q_l\, R_{1l}^3(\alpha_0)} - \tfrac{8}{3}\pi\mu\omega_0 c^3\cosh^2\alpha_0\sinh\alpha_0\right]e^{i\sigma t}, \end{aligned} \tag{50}$$

where a dash denotes differentiation with respect to α.

V. *Circular disk oscillating about its axis*

The motion due to rotatory oscillations of a circular disk about the normal through its centre can be obtained from the motion of the oblate spheroid as a limiting case. If we take the expression of the function $R_{1l}^3(\xi)$,

the 'radial solution' found in the case of the oblate spheroid, near $\xi = 0$, the required expression is an infinite series of associated Legendre functions. The appropriate form of the function (5), apart from a constant factor, may be written as $F_l(k,\xi)$, where, for even values of l,

$$F_l(k,\xi) = e^{i\pi l/2}\left\{\sum_{n=0}^{\infty} f_{2n}^l Q^1_{1+2n}(\sqrt{i}\,\xi)+f_{-2}^l\, Q^1_{-1}(\sqrt{i}\,\xi)+\right.$$
$$\left.+\sum_{n=2}^{\infty}\left(\frac{f_{-2n}^l}{\rho}\right)P^1_{2n-2}(\sqrt{i}\,\xi)-i\epsilon_l\, S^1_{1l}(k,\sqrt{i}\,\xi)\right\} \quad (51)$$

and
$$\epsilon_l = \tfrac{1}{6}\pi c f_0^l f_{-2}^l\left\{\frac{(\frac{1}{2}l)!}{[\frac{1}{2}(l+1)]!}\right\}^2,$$

while for odd values of l,

$$F_l(k,\xi) = e^{i\pi l/2}\left[\sum_{n=1}^{\infty} f_{2n-1}^l Q^1_{2n}(\sqrt{i}\,\xi)+f_{-1}^l\, Q^1_0(\sqrt{i}\,\xi)+\right.$$
$$\left.+\sum_{n=1}^{\infty}\left(\frac{f_{-2n-1}^l}{\rho}\right)P^1_{2n-1}(\sqrt{i}\,\xi)-i\epsilon_l\, S^1_{1l}(k,\sqrt{i}\,\xi)\right] \quad (52)$$

and
$$\epsilon_l = \tfrac{1}{40}\pi c^3 f_{+1}^l f_{-1}^l\left\{\frac{\{\frac{1}{2}(l-1)\}!}{\{\frac{1}{2}(l+2)\}!}\right\}^2.$$

Having proved that $A_l = 0$ for odd values of l, we shall be concerned with the expression in (51).

The expression for Ω is now given as

$$\Omega = c\cosh\alpha\sin\beta\sum_{l=0}^{\infty} A_{2l}\, F_{2l}(\alpha) S^1_{1,2l}(\beta)e^{i\sigma t}, \quad (53)$$

where
$$A_{2l} = -\frac{4\omega_0\, c\cosh\alpha_0}{3q_l\, F_{2l}(\alpha_0)}.$$

In the limiting case when $\alpha_0 \to 0$, A_{2l} takes the value

$$A_{2l} = -\frac{4\omega_0\, c}{3q_{2l}\, F_{2l}(0)}.$$

Proceeding as before, we get the couple on the oblate spheroid as

$$G = e^{i\sigma t}\left[\tfrac{32}{9}\mu\omega_0\, c^3\cosh^3\alpha_0\sum_{l=0}^{\infty}\left\{\frac{F'_{2l}(\alpha_0)}{q_{2l}\, F_{2l}(\alpha_0)}\right\}-\tfrac{8}{3}\pi\mu\omega_0\, c^3\cosh^2\alpha_0\sinh\alpha_0\right].$$

The couple in the case of the circular disk is given by the limit of G as $\alpha_0 \to 0$, viz.

$$\tfrac{32}{9}\pi\mu\omega_0\, c^3\sum_{l=0}^{\infty}\frac{F'_{2l}(0)}{q_{2l}\, F_{2l}(0)}e^{i\sigma t}. \quad (54)$$

The values of $F_{2l}(0)$ and $F'_{2l}(0)$ are given as

$$\left.\begin{aligned} F_{2l}(0) &= (-1)^l\pi^{\frac{1}{2}}\left[(1+2\epsilon_{2l}/\pi)\sum_{n=0}^{\infty}(-1)^n\frac{(n+\frac{1}{2})!}{n!}f_{2n}^{2l}\right] \\ F'_{2l}(0) &= (-1)^l\left[2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(n+1)!\,(-\frac{1}{2})!}{(n-\frac{1}{2})!}f_{2n}^{2l}-f_{-2}^{2l}+\right. \\ &\qquad\left.+4\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(n-\frac{1}{2})!}{(n-2)!\,(-\frac{1}{2})!}\left(\frac{f_{-2n}^{2l}}{\rho}\right)\right] \end{aligned}\right\}. \tag{55}$$

Part II. Longitudinal Oscillations

VI. *Sphere oscillating along a diameter*

The sphere is oscillating longitudinally along a diameter. The equations governing such a motion are

$$D^2\psi_1 = 0,$$

$$\left\{D^2-\frac{1}{\nu}\frac{\partial}{\partial t}\right\}\psi_2 = 0.$$

Taking the spherical polar coordinates R, θ, ϕ and writing $\psi_1 = \psi'_1 e^{i\sigma t}$, $\psi_2 = \psi'_2 e^{i\sigma t}$, the above equations reduce to

$$\frac{\partial^2\psi'_1}{\partial R^2}+\frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi'_1}{\partial\theta}\right) = 0 \tag{56}$$

and

$$\frac{\partial^2\psi'_2}{\partial R^2}+\frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi'_2}{\partial\theta}\right)+h^2\psi'_2 = 0, \tag{57}$$

where

$$h^2 = -i\sigma/\nu.$$

The boundary conditions are

$$\psi_{R=a} = -\tfrac{1}{2}w_0\,a^2\sin^2\theta\,e^{i\sigma t},$$

$$\left(\frac{\partial\psi}{\partial R}\right)_{R=a} = -w_0\,a\sin^2\theta\,e^{i\sigma t},$$

$$u = v = 0 \quad \text{at infinity},$$

where $w_0 e^{i\sigma t}$ is the velocity of the sphere along the diameter.

With the behaviour of the motion at infinity in view, the solution of the equations (56), (57) is

$$\psi'_1 = A\frac{\sin^2\theta}{R}, \tag{58}$$

$$\psi'_2 = Be^{-ihR}\left\{1+\frac{1}{ihR}\right\}\sin^2\theta. \tag{59}$$

Therefore

$$\psi = \left\{\frac{A}{r}+Be^{-ihR}\left(1+\frac{1}{ihR}\right)\right\}\sin^2\theta\,e^{i\sigma t}. \tag{60}$$

The boundary conditions give

$$A = \{\tfrac{3}{2}+\tfrac{3}{2}iha-\tfrac{1}{2}h^2a^2\}\frac{w_0 a}{h^2},$$

$$B = \frac{3}{2}\frac{w_0 a e^{iha}}{ih}.$$

The velocity components are given by

$$u = \frac{1}{R^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{1}{R\sin\theta}\frac{\partial\psi}{\partial R}, \qquad w = 0,$$

which when calculated agree with results already obtained by different methods (**1**, **2**).

VII. *Prolate spheroid oscillating along its axis of revolution*

As in Part I we introduce the prolate spheroidal coordinates α, β, γ such that

$$z+ir = c\cosh(\alpha+i\beta),$$

$$\phi = \gamma,$$

$$e_1 = e_2 = c\sqrt{(\cosh^2\alpha-\cos^2\beta)}, \qquad e_3 = c\sinh\alpha\sin\beta.$$

The equations of motion (7)–(8) are transformed into

$$\left\{\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\coth\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}\right\}\psi_1 = 0, \tag{61}$$

$$\left[\left\{\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\coth\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}\right\}-\frac{c^2(\cosh^2\alpha-\cos^2\beta)}{\nu}\frac{\partial}{\partial t}\right]\psi_2 = 0. \tag{62}$$

Writing $\psi_1 = \psi_1' e^{i\sigma t}$, $\psi = \psi_2' e^{i\sigma t}$, and $k^2 = \sigma/\nu$, equations (61)–(62) reduce to

$$\left\{\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\coth\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}\right\}\psi_1' = 0 \tag{63}$$

and

$$\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\coth\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}-ik^2c^2(\cosh^2\alpha-\cos^2\beta)\right]\psi_2' = 0. \tag{64}$$

Putting

$$\psi_1' = \sinh\alpha\sin\beta\,\theta(\alpha)\phi(\beta),$$

$$\xi = \sinh\alpha, \qquad \eta = \cos\beta,$$

equation (63) separates into the following two equations with λ as the separation constant:

$$\frac{d}{d\eta}\left\{(\eta^2-1)\frac{d\phi}{d\eta}\right\}+\left\{\lambda-\frac{1}{\eta^2-1}\right\}\phi = 0, \tag{65}$$

$$\frac{d}{d\xi}\left\{(\xi^2-1)\frac{d\theta}{d\xi}\right\}+\left\{\lambda-\frac{1}{\xi^2-1}\right\}\theta = 0, \tag{66}$$

which are the particular cases of the associated Legendre differential equation. Thus the appropriate solution of (63), keeping in view the boundary condition at infinity, is

$$\psi_1' = \sinh\alpha \sin\beta \sum_{n=0}^{\infty} A_n\, Q^1_{1+n}(\alpha) P^1_{1+n}(\beta). \tag{67}$$

The solution of (64), as already found in Part I, is

$$\psi_2' = \sinh\alpha \sin\beta \sum_{l=0}^{\infty} B_l\, R^3_{1l}(\alpha) S^1_{1l}(\beta), \tag{68}$$

where

$$S^1_{1l}(\beta) = \sum_{n=0,1}^{\infty}{}' d^l_n\, P^1_{1+n}(\beta)$$

and

$$R^3_{1l}(\surd i\, kc, \xi) = \frac{(\xi^2-1)^{\frac{1}{2}}\surd(\pi/2\surd i\, kc\xi^3) \sum\limits_{n=0,1}^{\infty}{}' i^{l-n} d^l_n (n+1)(n+2) K_{n+\frac{3}{2}}(\surd i\, kc\xi)}{\sum\limits_{n=0,1}^{\infty}{}' d^l_n (n+1)(n+2)}.$$

The complete solution of the problem is

$$\psi = \sinh\alpha \sin\beta \Big[\sum_{n=0}^{\infty} A_n\, Q^1_{1+n}(\alpha) P^1_{1+n}(\beta) + \sum_{l=0}^{\infty} B_l\, R^3_{1l}(\alpha) S^1_{1l}(\beta)\Big] e^{i\sigma t}. \tag{69}$$

10. Determination of the constants of integration A_n and B_l

The constants of integration A_n and B_l can be determined from the boundary conditions at the surface of the prolate spheroid, viz.

$$u = -\frac{w_0 \sinh\alpha_0 \cos\beta}{(\cosh^2\alpha_0 - \cos^2\beta)^{\frac{1}{2}}} e^{i\sigma t} = \left\{\frac{1}{e_2 e_3}\frac{\partial\psi}{\partial\beta}\right\}_{\alpha=\alpha_0},$$

$$v = \frac{w_0 \cosh\alpha_0 \sin\beta}{(\cosh^2\alpha_0 - \cos^2\beta)^{\frac{1}{2}}} e^{i\sigma t} = -\left\{\frac{1}{e_1 e_3}\frac{\partial\psi}{\partial\alpha}\right\}_{\alpha=\alpha_0},$$

which require

$$(\psi)_{\alpha=\alpha_0} = -\tfrac{1}{2} w_0 c^2 \sinh^2\alpha_0 \sin^2\beta\, e^{i\sigma t},$$

$$\left(\frac{\partial\psi}{\partial\alpha}\right)_{\alpha=\alpha_0} = -w_0 c^2 \sinh\alpha_0 \cosh\alpha_0 \sin^2\beta\, e^{i\sigma t},$$

where $w_0 e^{i\sigma t}$ is the velocity of the body along the axis of symmetry.

Now ψ and $\partial\psi/\partial\alpha$ are given as

$$\psi = \Big[\sum_{n=0}^{\infty} A_n \sinh\alpha\, Q^1_{1+n}(\alpha) P^1_{1+n}(\beta) \sin\beta + \sum_{l=0}^{\infty} B_l \sinh\alpha\, R^3_{1l}(\alpha) S^1_{1l}(\beta) \sin\beta\Big] e^{i\sigma t}, \tag{70}$$

$$\frac{\partial\psi}{\partial\alpha} = \Big[\sum_{n=0}^{\infty} A_n \{\cosh\alpha\, Q^1_{1+n}(\alpha) + \sinh\alpha\, Q^{1\prime}_{1+n}(\alpha)\} P^1_{1+n}(\beta) \sin\beta +$$

$$+ \sum_{l=0}^{\infty} B_l \{\cosh\alpha\, R^3_{1l}(\alpha) + \sinh\alpha\, R^{3\prime}_{1l}(\alpha)\} S^1_{1l}(\beta) \sin\beta\Big] e^{i\sigma t}, \tag{71}$$

where a dash denotes differentiation with respect to α.

We rewrite these expressions in forms which are suitable for fitting the boundary conditions at the surface of the body. For this purpose we expand the right-hand sides of (70) and (71) into associated Legendre functions. Let us introduce functions $f_n(\alpha)$ and $F_{l,n}(\alpha)$ such that

$$\left.\begin{aligned} f_n(\alpha) &= Q^1_{1+n}(\alpha)\sinh\alpha \\ \text{and}\qquad \sum_{n=0}^{\infty} F_{l,n}(\alpha)P^1_{1+n}(\beta) &= \sinh\alpha\, R^3_{1l}(\alpha)S^1_{1l}(\beta) \end{aligned}\right\}. \tag{72}$$

Equating the coefficients of the associated Legendre functions of the same order on both sides, we get

$$F_{l,n}(\alpha) = \sinh\alpha\, R^3_{1l}(\alpha)d^l_n. \tag{73}$$

The boundary conditions take the form

$$\left.\begin{aligned} \tfrac{1}{2}w_0\, c^2\sinh^2\alpha_0\, P^1_1(\beta) &= \sum_{n=0}^{\infty}\Big[A_n f_n(\alpha_0)+\sum_{l=0}^{\infty} B_l\, F_{l,n}(\alpha_0)\Big]P^1_{1+n}(\beta) \\ w_0\, c^2\sinh\alpha_0\cosh\alpha_0\, P^1_1(\beta) &= \sum_{n=0}^{\infty}\Big[A_n f'_n(\alpha_0)+\sum_{l=0}^{\infty} B_l\, F'_{l,n}(\alpha_0)\Big]P^1_{1+n}(\beta) \end{aligned}\right\}, \tag{74}$$

which yield

$$\left.\begin{aligned} A_n f_n(\alpha_0)+\sum_{l=0}^{\infty} B_l\, F_{l,n}(\alpha_0) &= \begin{cases} \tfrac{1}{2}w_0\, c^2\sinh^2\alpha_0 & \text{for } n=0 \\ 0 & \text{for } n>0 \end{cases} \\ A_n f'_n(\alpha_0)+\sum_{l=0}^{\infty} B_l\, F'_{l,n}(\alpha_0) &= \begin{cases} w_0\, c^2\sinh\alpha_0\cosh\alpha_0 & \text{for } n=0 \\ 0 & \text{for } n>0 \end{cases} \end{aligned}\right\}. \tag{75}$$

Eliminating A_n, we get

$$\begin{aligned} &\sum B_l\{F_{l,n}(\alpha_0)f'_n(\alpha_0)-F'_{l,n}(\alpha_0)f_n(\alpha_0)\} \\ &\qquad = \begin{cases} w_0\, c^2\sinh\alpha_0\{\tfrac{1}{2}\sinh\alpha_0 f'_n(\alpha_0)-\cosh\alpha_0 f_n(\alpha_0)\} & \text{for } n=0, \\ 0 & \text{for } n>0. \end{cases} \end{aligned} \tag{76}$$

The B_l can be determined by solving this system of simultaneous linear algebraic equations. Tomotika and Aoi (**6**) have made some suggestions for the solution of a similar set of equations. In our case it is not possible to give similar expansions at this stage as the values of the coefficients d^l_n which have been tabulated to date are of little use to us here.

The stream function is thus completely determined.

VIII. *Oblate spheroid oscillating along its axis of revolution*

For an oblate spheroid the system of coordinates is

$$z+ir = c\sinh(\alpha+i\beta),$$

$$\phi = \gamma,$$

$$e_1 = e_2 = c(\sinh^2\alpha+\cos^2\beta)^{\frac{1}{2}}, \qquad e_3 = c\cosh\alpha\sin\beta.$$

The equations of motion when transformed into these coordinates become

$$\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\tanh\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}\right)\psi_1 = 0, \tag{77}$$

$$\left\{\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\tanh\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}-\frac{c^2}{\nu}(\sinh^2\alpha+\cos^2\beta)\frac{\partial}{\partial t}\right\}\psi_2 = 0. \tag{78}$$

Writing $\psi_1 = \psi_1' e^{i\sigma t}$, $\psi_2 = \psi_2' e^{i\sigma t}$, and $k^2 = \sigma/\nu$, we get

$$\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\tanh\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}\right)\psi_1' = 0, \tag{79}$$

$$\left\{\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\tanh\alpha\frac{\partial}{\partial\alpha}-\cot\beta\frac{\partial}{\partial\beta}-ik^2c^2(\sinh^2\alpha+\cos^2\beta)\right\}\psi_2' = 0. \tag{80}$$

To solve equation (79) we put

$$\psi_1' = \cosh\alpha\sin\beta\,\theta(\alpha)\phi(\beta),$$
$$\sinh\alpha = \xi, \qquad \cos\beta = \eta.$$

The equation separates into

$$\frac{d}{d\eta}\left[(\eta^2-1)\frac{d\phi}{d\eta}\right]+\left[\lambda-\frac{1}{\eta^2-1}\right]\phi = 0 \tag{81}$$

and

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d\theta}{d\xi}\right]+\left[\lambda+\frac{1}{\xi^2+1}\right]\theta = 0, \tag{82}$$

where λ is a constant of separation.

The required solution of (79) is therefore

$$\psi_1' = \cosh\alpha\sin\beta\sum_{l=0}^{\infty}A_n\,Q_{1+n}^1(i\sinh\alpha)P_{1+n}^1(\cos\beta). \tag{83}$$

The solution of (80), as already obtained in Part I, is

$$\psi_2' = \cosh\alpha\sin\beta\sum_{l=0}^{\infty}B_l\,R_{1l}^3(\alpha)S_{1l}^1(\beta). \tag{84}$$

The superposition of these two solutions gives the complete solution

$$\psi = \cosh\alpha\sin\beta\left[\sum_{n=0}^{\infty}A_n\,Q_{1+n}^1(\alpha)P_{1+n}^1(\beta)+\sum_{l=0}^{\infty}B_l\,R_{1l}^3(\alpha)S_{1l}^1(\beta)\right]e^{i\sigma t}. \tag{85}$$

11. Determination of the constants of integration A_n and B_l

The boundary conditions, in this case, are

$$u = -\frac{w_0\cosh\alpha_0\cos\beta}{(\sinh^2\alpha_0+\cos^2\beta)^{\frac{1}{2}}}e^{i\sigma t} = \left(\frac{1}{e_2e_3}\frac{\partial\psi}{\partial\beta}\right)_{\alpha=\alpha_0},$$

$$v = -\frac{w_0\sinh\alpha_0\sin\beta}{(\sinh^2\alpha_0+\cos^2\beta)^{\frac{1}{2}}}e^{i\sigma t} = -\left(\frac{1}{e_1e_3}\frac{\partial\psi}{\partial\alpha}\right)_{\alpha=\alpha_0},$$

which reduce to

$$\left.\begin{aligned}(\psi)_{\alpha=\alpha_0} &= -\tfrac{1}{2}w_0\,c^2\cosh^2\alpha_0\sin^2\beta\,e^{i\sigma t}\\ \left(\frac{\partial\psi}{\partial\alpha}\right)_{\alpha=\alpha_0} &= -w_0\,c^2\cosh\alpha_0\sinh\alpha_0\sin^2\beta\,e^{i\sigma t}\end{aligned}\right\}. \tag{86}$$

Proceeding as in the case of the prolate spheroid, we introduce functions $g_n(\alpha)$ and $G_{l,n}(\alpha)$ such that

$$\left.\begin{aligned}g_n(\alpha) &= \cosh\alpha\,Q^1_{1+n}(\alpha)\\ \sum_{n=0}^{\infty} G_{l,n}(\alpha)P^1_{1+n}(\beta) &= \cosh\alpha\,R^3_{1l}(\alpha)S^1_{1l}(\beta)\end{aligned}\right\}. \tag{87}$$

Equating the coefficients of the associated Legendre functions of the same order, we get

$$G_{l,n}(\alpha) = \cosh\alpha\,R^3_{1l}(\alpha)f'_n. \tag{88}$$

The boundary conditions take the form

$$\left.\begin{aligned}\tfrac{1}{2}w_0\,c^2\cosh^2\alpha_0\,P^1_1(\beta) &= \sum_{n=0}^{\infty}\Big[A_n\,g_n(\alpha_0)+\sum_{l=0}^{\infty}B_l\,G_{l,n}(\alpha_0)\Big]P^1_{1+n}(\beta)\\ w_0\,c^2\cosh\alpha_0\sinh\alpha_0\,P^1_1(\beta) &= \sum_{n=0}^{\infty}\Big[A_n\,g'_n(\alpha_0)+\sum_{l=0}^{\infty}B_l\,G'_{l,n}(\alpha_0)\Big]P^1_{1+n}(\beta)\end{aligned}\right\}, \tag{89}$$

where dashes denote differentiation with respect to α.

Equations (89) yield

$$\left.\begin{aligned}A_n\,g_n(\alpha_0)+\sum_{l=0}^{\infty}B_l\,G_{l,n}(\alpha_0) &= \begin{cases}\tfrac{1}{2}w_0\,c^2\cosh^2\alpha_0 & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}\\ A_n\,g'_n(\alpha_0)+\sum_{l=0}^{\infty}B_l\,G'_{l,n}(\alpha_0) &= \begin{cases}w_0\,c^2\cosh\alpha_0\sinh\alpha_0 & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}\end{aligned}\right\}. \tag{90}$$

Eliminating A_n, we get a set of algebraic equations

$$\begin{aligned}&\sum B_l\{G_{l,n}(\alpha_0)g'_n(\alpha_0)-G'_{l,n}(\alpha_0)g_n(\alpha_0)\}\\ &\qquad= \begin{cases}w_0\,c^2\cosh\alpha_0[\tfrac{1}{2}\cosh\alpha_0\,g'_n(\alpha_0)-\sinh\alpha_0\,g_n(\alpha_0)] & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}\end{aligned} \tag{91}$$

to determine the values of the constants B_l. These equations involve the coefficients f^l_n, and those so far tabulated only are of no use to us here.

IX. *Circular disk oscillating along the normal axis at its centre*

For deducing the stream function for the motion of the disk as a limiting case of the oblate spheroid we take the expression of the function $R^3_{1l}(\xi)$ for values of ξ near $\xi = 0$. The required expression, as in Part I, is $F_l(k,\xi)$,

which, apart from a constant factor, is given for even values of l as

$$F_l(k,\xi) = e^{i\pi l/2}\left\{\sum_{n=0}^{\infty} f_{2n}^l\, Q_{1+2n}^1(\sqrt{i}\,\xi)+f_{-2}^l\, Q_{-1}^1(\sqrt{i}\,\xi)+ \right.$$
$$\left. +\sum_{n=2}^{\infty}\left(\frac{f_{-2n}^l}{\rho}\right)P_{2n-1}^1(\sqrt{i}\,\xi)-i\epsilon_l\, S_{1l}^1(k,\sqrt{i}\,\xi)\right\},$$

while, for odd values of l,

$$F_l(k,\xi) = e^{i\pi l/2}\left\{\sum_{n=1}^{\infty} f_{2n-1}^l\, Q_{2n}^1(\sqrt{i}\,\xi)+f_{-1}^l\, Q_0^1(\sqrt{i}\,\xi)+ \right.$$
$$\left. +\sum_{n=1}^{\infty}\left(\frac{f_{-2n-1}^l}{\rho}\right)P_{2n-1}^1(\sqrt{i}\,\xi)-i\epsilon_l\, S_{1l}^1(k,\sqrt{i}\,\xi)\right\}.$$

The expression for the stream function for an oblate spheroid is now given as

$$\psi = \cosh\alpha\sin\beta\Big[\sum_{n=0}^{\infty} A_n\, Q_{1+n}^1(\alpha)P_{1+n}^1(\beta)+\sum_{l=0}^{\infty} B_l\, F_l(\alpha)S_{1l}^1(\beta)\Big]e^{i\sigma t}. \qquad (92)$$

A_n and B_l are determined by the equations

$$\left.\begin{aligned} A_n\, g_n(\alpha_0)+\sum_{l=0}^{\infty} B_l\, H_{l,n}(\alpha_0) &= \begin{cases}\tfrac{1}{2}w_0\, c^2\cosh^2\alpha_0 & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}\\ A_n\, g_n'(\alpha_0)+\sum_{l=0}^{\infty} B_l\, H_{l,n}'(\alpha_0) &= \begin{cases}w_0\, c^2\cosh\alpha_0\sinh\alpha_0 & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}\end{aligned}\right\}, \qquad (93)$$

where the function $H_{l,n}(\alpha)$, being defined in the same way as $G_{l,n}(\alpha)$, is

$$H_{l,n}(\alpha) = \cosh\alpha\, F_l(\alpha)d_n^l. \qquad (94)$$

In the limiting case when $\alpha_0\to 0$, A_n and B_l are given by

$$\left.\begin{aligned} A_n\, g_n(0)+\sum_{l=0}^{\infty} B_l\, H_{l,n}(0) &= \begin{cases}\tfrac{1}{2}w_0\, c^2 & \text{for } n=0\\ 0 & \text{for } n>0\end{cases}\\ A_n\, g_n'(0)+\sum_{l=0}^{\infty} B_l\, H_{l,n}'(0) &= 0 \qquad \text{for all } n\end{aligned}\right\}. \qquad (95)$$

Elimination of A_n yields us a set of algebraic equations

$$\sum_{l=0}^{\infty} B_l\{H_{l,n}(0)g_n'(0)-H_{l,n}'(0)g_n(0)\} = \begin{cases}\tfrac{1}{2}w_0\, c^2 g_n'(0) & \text{for } n=0\\ 0 & \text{for } n>0\end{cases} \qquad (96)$$

for the determination of the B_l. They are similar to the equations (91).

In conclusion I wish to thank Professor B. R. Seth for his guidance in the course of the preparation of this paper. My thanks are also due to Professor L. Rosenhead and Mr. L. Sowerby for their helpful suggestions.

REFERENCES

1. R. P. KANWAL, 'Generalized periodic singular points with applications to flow problems' (to be published).
2. H. LAMB, *Hydrodynamics* (Dover, New York, 1945), pp. 642–4.
3. J. A. STRATTON, P. M. MORSE, L. J. CHU, and R. A. HUTNER, *Elliptic Cylinder and Spheroidal Wave Functions* (New York, 1941).
4. S. GOLDSTEIN, *Modern Developments in Fluid Dynamics* (Oxford, 1938), pp. 114–15.
5. L. SOWERBY, 'Couple on a rotating spheroid in a slow stream', *Proc. Camb. Phil. Soc.* **49** (1935), 327–32.
6. S. TOMOTIKA and T. AOI, 'The steady flow of a viscous fluid past an elliptic cylinder and a flat plate at small Reynolds numbers', *Quart. J. Mech. and Applied Math.* **6** (1953), 295.

ON THE DIFFUSION OF LOAD FROM A STIFFENER INTO A SHEET

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SUMMARY

A rigorous solution is obtained for the diffusion of load from a semi-infinite stiffener into an infinite or a semi-infinite sheet. A Mellin transformation, applied to the basic singular integro-differential equation of the problem, yields a difference equation in a strip of the complex plane which is solved by means of Laplace transforms. The final solution is obtained in the form of an inverse Mellin integral which is evaluated by contour integration.

1. Introduction

THE diffusion of load from a stiffener into a sheet is a fundamental problem of aircraft stress analysis. It occurs in many forms and it has been discussed by many authors, but so far very few if any rigorous solutions have been obtained. It is the object of the present paper to give the *rigorous* solution of this problem for an *infinite* or *semi-infinite sheet* with a *semi-infinite stiffener* of *constant* cross-section.

A semi-infinite sheet with a semi-infinite edge stiffener of constant cross-section has been investigated by Buell (**1**) by means of a complex stress function for the sheet. Although his solution is entirely adequate for all practical purposes, it is not rigorous because the convergence of his process of successive approximations to an infinite system of equations has not been established. Benscoter (**2**) has discussed an infinite sheet with a finite stiffener, starting from the integro-differential equation for the shear stress between stiffener and sheet. He observed that this equation is formally identical with Prandtl's equation for the aerodynamic load distribution over a wing of finite span, and therefore amenable to the same (approximate) methods of solution. Our rigorous solution for a semi-infinite stiffener is also based on this equation—which, apart from a different scale factor, also applies to a semi-infinite sheet with an edge stiffener—but the method of solution, i.e. the application of Mellin transforms, is quite different. It may be observed that our solution is likewise applicable to the aerodynamic load distribution over a semi-infinite wing.

2. The integro-differential equation

We consider an infinite or semi-infinite sheet (Figs. 1 and 2), loaded by

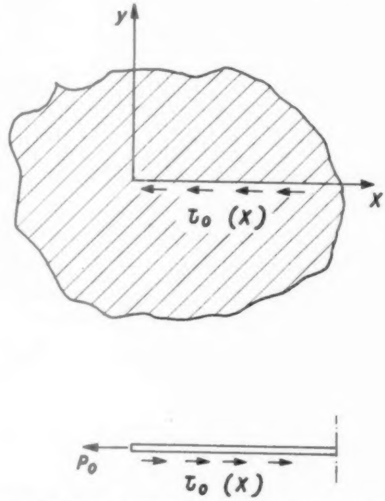

FIG. 1. Infinite sheet with semi-infinite stiffener.

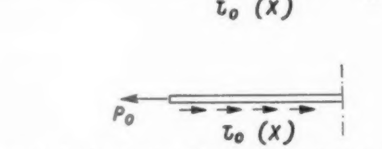

FIG. 2. Semi-infinite sheet with semi-infinite edge stiffener.

a system of continuously distributed forces over the positive x-axis and directed along the negative x-axis; these loads are applied by the stiffener and their reactions act on the stiffener together with the end load P_0. The direct strain ϵ_x along the x-axis in the sheet in the case of an *infinite sheet* is given by (**2**)

$$(\epsilon_x)_{y=0} = -\frac{(3-\nu)(1+\nu)}{4\pi E}\int\limits_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi, \tag{1}$$

where the integral is defined as Cauchy's principal value, and in the case of a *semi-infinite* sheet by (3)

$$(\epsilon_x)_{y=0} = -\frac{2}{\pi E}\int\limits_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi, \tag{2}$$

where E is Young's modulus and ν is Poisson's ratio. The axial strain in the stiffener is given by

$$\epsilon_{xs} = \frac{1}{E_s A_s}\Big[P_0 - h\int\limits_0^x \tau_0(\xi)\,d\xi\Big], \tag{3}$$

where E_s is Young's modulus for the stiffener, A_s is the area of the cross-section, and h is the sheet thickness. Equating expressions (1) and (3) (or (2) and (3) for a semi-infinite sheet) yields the integro-differential equation for the shear stress $\tau_0(x)$ between stiffener and sheet.

The integro-differential equation may be written in non-dimensional form by the following substitutions for an *infinite* sheet

$$\left.\begin{aligned} x &= \tfrac{1}{8}(3-\nu)(1+\nu)\frac{E_s A_s}{Eh}\bar{x}, \qquad \xi = \tfrac{1}{8}(3-\nu)(1+\nu)\frac{E_s A_s}{Eh}\bar{\xi} \\ \tau_0 &= \frac{8}{(3-\nu)(1+\nu)}\frac{P_0 E}{E_s A_s}\bar{\tau}_0 \end{aligned}\right\}, \tag{4}$$

the corresponding substitutions for a *semi-infinite* sheet are

$$x = \frac{E_s A_s}{Eh}\bar{x}, \qquad \xi = \frac{E_s A_s}{Eh}\bar{\xi}, \qquad \tau_0 = \frac{P_0 E}{E_s A_s}\bar{\tau}_0. \tag{5}$$

Dropping the bars over the non-dimensional variables, the basic integro-differential equation now reads

$$-\frac{2}{\pi}\int\limits_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi + \int\limits_0^x \tau_0(\xi)\,d\xi - 1 = 0. \tag{6}$$

The additional requirement that all stresses vanish at infinity is expressed by the equation

$$\int\limits_0^\infty \tau_0(\xi)\,d\xi = 1. \tag{7}$$

3. Solution of the integro-differential equation

We multiply both sides of (6) by x^{s-1} and integrate from $x = 0$ to $x = \infty$, obtaining

$$-\frac{2}{\pi}\int\limits_0^\infty x^{s-1}\,dx\int\limits_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi + \int\limits_0^\infty x^{s-1}\,dx\Big[\int\limits_0^x \tau_0(\xi)\,d\xi - 1\Big] = 0. \tag{8}$$

Formal inversion of the order of integration in the first term and integration by parts of the second term results in

$$-\frac{2}{\pi}\int\limits_0^\infty \xi^{s-1}\tau_0(\xi)\,d\xi \int\limits_0^\infty \frac{\eta^{s-1}}{1-\eta}\,d\eta-\frac{1}{s}\int\limits_0^\infty x^s\tau_0(x)\,dx = 0. \tag{9}$$

Introducing the Mellin transforms (**4**)

$$T_0(s) = \int\limits_0^\infty x^{s-1}\tau_0(x)\,dx, \tag{10}$$

$$H(s) = \int\limits_0^\infty \frac{x^{s-1}}{1-x}\,dx, \tag{11}$$

this equation is written in the form of a difference equation with variable coefficients for the transform $T_0(s)$

$$T_0(s+1) = -\frac{2}{\pi}sH(s)T_0(s). \tag{12}$$

The foregoing formal analysis may be justified if the following assumptions are made on the behaviour of $\tau_0(x)$:

(*a*) $\tau_0(x)x^{\lambda-1}$ is $L(0,\infty)$ for all λ in a range $b<\lambda<1$, where $0<b<1$;

(*b*) $|\tau_0(\xi)-\tau_0(x)| \leqslant F(x)|\xi-x|$ for all $|\xi-x|<\epsilon x$, where $\epsilon<1$ is an arbitrarily small positive number and $F(x)x^\lambda$ is $L(0,\infty)$ for all λ in the range $b<\lambda<1$;

(*c*) $$x^\lambda\Big[\int\limits_0^x \tau_0(\xi)\,d\xi-1\Big] \to 0 \quad \text{and} \quad x^\lambda\int\limits_x^\infty |\tau_0(\xi)|\,d\xi \to 0$$

for $x\to\infty$ and $b<\lambda<1$.

These assumptions may be verified once the solution has been obtained.

It is now obvious that integration by parts of the second term in (8) is justified by assumption (*c*) in the strip $b<\operatorname{re}s<1$. In order to justify the inversion of the order of integration in the first term we write†

$$\begin{aligned}\int\limits_0^\infty x^{s-1}\,dx\int\limits_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi &= \lim_{A\to\infty}\Bigg\{\int\limits_0^A x^{s-1}\,dx\int\limits_0^{x(1-\epsilon)}\frac{\tau_0(\xi)}{\xi-x}\,d\xi+\int\limits_0^A x^{s-1}\,dx\int\limits_{x(1-\epsilon)}^{x(1+\epsilon)}\frac{\tau_0(\xi)}{\xi-x}\,d\xi+\\ &\quad+\int\limits_0^A x^{s-1}\,dx\int\limits_{x(1+\epsilon)}^{A(1+\epsilon)}\frac{\tau_0(\xi)}{\xi-x}\,d\xi+\int\limits_0^A x^{s-1}\,dx\int\limits_{A(1+\epsilon)}^\infty\frac{\tau_0(\xi)}{\xi-x}\,d\xi\Bigg\},\end{aligned} \tag{13}$$

† The author is indebted to his colleague Professor S. C. van Veen for helpful criticism of this inversion

where $0 < \epsilon < 1$. Bearing in mind that the inner integral in (8) is defined as a principal value, the inner integral in the second term is also defined as a principal value, and it follows from assumption (b) that the second term in (13) is bounded and of order ϵ in the strip $b < \operatorname{re} s < 1$. The last term tends to zero for $A \to \infty$ in the strip $b < \operatorname{re} s < 1$ on account of the inequality

$$\left|\int_0^A x^{s-1}\,dx \int_{A(1+\epsilon)}^{\infty} \frac{\tau_0(\xi)}{\xi - x}\,d\xi\right| < \int_0^A |x^{s-1}|\,dx \int_{A(1+\epsilon)}^{\infty} |\tau_0(\xi)|\,\frac{d\xi}{A\epsilon}$$

$$< \frac{1}{A\epsilon}\,\frac{1}{\sigma}\,A^{\sigma} \int_A^{\infty} |\tau_0(\xi)|\,d\xi, \tag{14}$$

where $\sigma = \operatorname{re} s$. In the two remaining terms the order of integration may be inverted (cf. Fig. 3). Putting $x/\xi = \eta$, we write

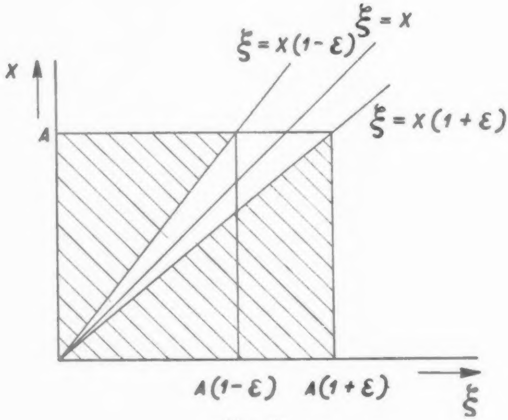

FIG. 3.

$$\lim_{A\to\infty}\left\{\int_0^A x^{s-1}\,dx \int_0^{x(1-\epsilon)} \frac{\tau_0(\xi)}{\xi - x}\,d\xi + \int_0^A x^{s-1}\,dx \int_{x(1+\epsilon)}^{A(1+\epsilon)} \frac{\tau_0(\xi)}{\xi - x}\,d\xi\right\}$$

$$= \lim_{A\to\infty}\left\{\int_0^{A(1-\epsilon)} \tau_0(\xi)\,d\xi \int_{\xi/(1-\epsilon)}^{A} \frac{x^{s-1}}{\xi - x}\,dx + \int_0^{A(1+\epsilon)} \tau_0(\xi)\,d\xi \int_0^{\xi/(1+\epsilon)} \frac{x^{s-1}}{\xi - x}\,dx\right\}$$

$$= \lim_{A\to\infty}\left\{\int_0^{A(1-\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_{1/(1-\epsilon)}^{A/\xi} \frac{\eta^{s-1}}{1-\eta}\,d\eta + \int_0^{A(1+\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_0^{1/(1+\epsilon)} \frac{\eta^{s-1}}{1-\eta}\,d\eta\right\}. \tag{15}$$

The limit of the second term may be written down immediately. In order to obtain the limit of the first term, we write, with $0 < B < A$,

$$\int_0^{A(1-\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_{1/(1-\epsilon)}^{A/\xi} \frac{\eta^{s-1}}{1-\eta}\,d\eta$$

$$= \int_0^{B(1-\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_{1/(1-\epsilon)}^{A/\xi} \frac{\eta^{s-1}}{1-\eta}\,d\eta + \int_{B(1-\epsilon)}^{A(1-\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_{1/(1-\epsilon)}^{A/\xi} \frac{\eta^{s-1}}{1-\eta}\,d\eta.$$

Because $A/\xi \geqslant 1/(1-\epsilon)$, the inner integral exists, and it remains bounded for $A \to \infty$ in the strip $b < \operatorname{re} s < 1$. Therefore

$$\lim_{A\to\infty} \int_0^{A(1-\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_{1/(1-\epsilon)}^{A/\xi} \frac{\eta^{s-1}}{1-\eta}\,d\eta$$

$$= \int_0^{B(1-\epsilon)} \xi^{s-1}\tau_0(\xi)\,d\xi \int_{1/(1-\epsilon)}^{\infty} \frac{\eta^{s-1}}{1-\eta}\,d\eta + \text{remainder term},$$

where

$$|\text{remainder term}| < \int_{B(1-\epsilon)}^{\infty} |\xi^{s-1}\tau_0(\xi)|\,d\xi \int_{1/(1-\epsilon)}^{\infty} \left|\frac{\eta^{s-1}}{1-\eta}\right| d\eta.$$

If we now let B tend to infinity, the remainder term tends to zero. Hence we may write (13) in the form

$$\int_0^\infty x^{s-1}\,dx \int_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi$$

$$= \int_0^\infty \xi^{s-1}\tau_0(\xi)\,d\xi \left\{ \int_0^{1/(1+\epsilon)} \frac{\eta^{s-1}}{1-\eta}\,d\eta + \int_{1/(1-\epsilon)}^{\infty} \frac{\eta^{s-1}}{1-\eta}\,d\eta \right\} + O(\epsilon)$$

$$= \int_0^\infty \xi^{s-1}\tau_0(\xi)\,d\xi \left\{ \int_0^{1-\epsilon} \frac{\eta^{s-1}}{1-\eta}\,d\eta + \int_{1+\epsilon}^{\infty} \frac{\eta^{s-1}}{1-\eta}\,d\eta \right\} + O(\epsilon). \qquad (16)$$

This result holds for ϵ arbitrarily small. If we let ϵ tend to zero, we obtain, for $b < \operatorname{re} s < 1$,

$$\int_0^\infty x^{s-1}\,dx \int_0^\infty \frac{\tau_0(\xi)}{\xi-x}\,d\xi = \int_0^\infty \xi^{s-1}\tau_0(\xi)\,d\xi \int_0^\infty \frac{\eta^{s-1}}{1-\eta}\,d\eta = T_0(s)H(s), \qquad (17)$$

where the integral $H(s)$ is again defined by Cauchy's principal value.

In order to evaluate the integral $H(s)$, we write

$$H(s) = \int_0^\infty \frac{\eta^{s-1}}{1-\eta}\,d\eta = \lim_{\epsilon\to 0}\left\{ \int_0^{1-\epsilon} \frac{\eta^{s-1}}{1-\eta}\,d\eta + \int_{1+\epsilon}^{\infty} \frac{\eta^{s-1}}{1-\eta}\,d\eta \right\}$$

and change the variable in the second integral into $\xi = 1/\eta$:

$$H(s) = \lim_{\epsilon\to 0}\left\{\int\limits_0^{1-\epsilon} \frac{\eta^{s-1}}{1-\eta}\,d\eta - \int\limits_0^{1/(1+\epsilon)} \frac{\zeta^{-s}}{1-\zeta}\,d\zeta\right\}$$

$$= \lim_{\epsilon\to 0}\left\{\int\limits_0^{1-\epsilon} \frac{\eta^{s-1}-\eta^{-s}}{1-\eta}\,d\eta - \int\limits_{1-\epsilon}^{1/(1+\epsilon)} \frac{\zeta^{-s}}{1-\zeta}\,d\zeta\right\} = \int\limits_0^1 \frac{\eta^{s-1}-\eta^{-s}}{1-\eta}\,d\eta. \tag{18}$$

Applying Legendre's formula (**5**, p. 260)

$$\int\limits_0^1 \frac{x^{s-1}-1}{x-1}\,dx = \frac{d\log\Gamma(s)}{ds}+\gamma, \tag{19}$$

where γ is the Euler–Mascheroni constant ($= 0{\cdot}5772157...$), we obtain finally

$$H(s) = -\frac{d\log\Gamma(s)}{ds} - \frac{d\log\Gamma(1-s)}{ds} = \pi\cot\pi s, \tag{20}$$

holding for $0 < \operatorname{re} s < 1$.

The difference equation (12) may now be written in the explicit form

$$T_0(s+1) = -2s\cot\pi s\,T_0(s), \tag{21}$$

holding for $b < \operatorname{re} s < 1$. The solution of this equation has to satisfy the condition that $T_0(s)$ is acceptable as the Mellin transform of $\tau_0(x)$, i.e. $T_0(s) \to 0$ for $|s| \to \infty$ in the strip $b < \operatorname{re} s < 1$. The solution obviously contains an arbitrary constant factor which may be determined from the 'boundary' condition (7), viz. $T_0(1) = 1$.

It may now be observed that $T_0(s)$, which is regular in the strip

$$b < \operatorname{re} s < 1$$

on account of assumption (a), is also regular in the wider strip

$$b < \operatorname{re} s < 2.$$

This statement is proved by considering the inverse transform in the second term of (6),

$$\int\limits_0^x \tau_0(\xi)\,d\xi - 1 = -\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{1}{s}T_0(s+1)x^{-s}\,ds,$$

where $b < c < 1$. On the other hand, we may write

$$\int\limits_0^x \tau_0(\xi)\,d\xi = \int\limits_0^x \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} T_0(s)\xi^{-s}\,ds = -\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{1}{s-1}T_0(s)x^{-s+1}\,ds.$$

Writing

$$1 = \frac{1}{2\pi i}\int\limits_{c+1-i\infty}^{c+1+i\infty} \frac{1}{s-1}x^{-s+1}\,ds - \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{1}{s-1}x^{-s+1}\,ds,$$

we may obtain the equation

$$-\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{1}{s-1}[T_0(s)-1]x^{-s+1}\,ds = -\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{1}{s}T_0(s+1)x^{-s}\,ds+$$
$$+\frac{1}{2\pi i}\int\limits_{c+1-i\infty}^{c+1+i\infty}\frac{1}{s-1}x^{-s+1}\,ds.$$

Writing $s+1 = s'$ in the first integral in the right-hand member and $s = s'$ in the second integral, this equation becomes

$$\int\limits_{c-i\infty}^{c+i\infty}\frac{1}{s-1}[T_0(s)-1]x^{-s+1}\,ds = \int\limits_{c+1-i\infty}^{c+1+i\infty}\frac{1}{s'-1}[T_0(s')-1]x^{-s'+1}\,ds'.$$

From a general theorem† given by Titchmarsh (**4**, art. 9.9) it now follows that $\frac{1}{s-1}[T_0(s)-1]$ is regular in the strip $c \leqslant \operatorname{re} s \leqslant c+1$ and the proof that $T_0(s)$ is regular in the strip $b < \operatorname{re} s < 2$ has been completed.

Finally, it may be stated here that the problem may also be attacked without recourse to the integro-differential equation (6). The state of stress in the sheet is governed by a stress function which satisfies the biharmonic equation. Applying the Mellin transform to this biharmonic equation and its boundary conditions ultimately results in the same difference equation (21) for the Mellin transform of the shear stress between stiffener and sheet. This alternative approach may also be used in similar problems where the kernel of the integro-differential equation is not known beforehand, e.g. for a wedge-shaped sheet with a semi-infinite stiffener. We may return to this more general problem in a later paper.

4. Solution of difference equation (21)

In order to simplify (21) we write

$$T_0(s) = 2^s\Gamma(s)\frac{1}{\sin\frac{1}{2}\pi s}y(s). \tag{22}$$

Substitution into (21) yields the equation

$$y(s+1) = \frac{\cos\pi s}{\cos\pi s-1}y(s) \tag{23}$$

with the 'boundary' condition $y(1) = \frac{1}{2}$. The factor $2^s\Gamma(s)$ in (22) has been introduced for obvious reasons. The factor $(\sin\frac{1}{2}\pi s)^{-1}$ serves to make the

† Titchmarsh formulates his theorem for Fourier transforms; its analogue for Mellin transforms is then nearly obvious.

cofactor of $y(s)$ in (23) tend to 1 as $\operatorname{im} s \to \pm\infty$ in the strip $b < \operatorname{re} s < 1$. Taking the logarithmic derivative of (23), we obtain

$$x(s+1)-x(s) = \frac{\pi \sin \pi s}{\cos \pi s(\cos \pi s-1)} = f(s), \tag{24}$$

where
$$x(s) = \frac{1}{y(s)}\frac{dy(s)}{ds}. \tag{25}$$

Equation (24) is much simpler than (21) because it is a difference equation with *constant* coefficients and it is solved by Laplace transforms.† Assuming that $x(\sigma+i\tau)e^{\mu\tau}$ is $L(-\infty,\infty)$ in the strip $b < \sigma = \operatorname{re} s < 1$ for all μ in the range $-\pi < -d < \mu < d < \pi$, we may introduce the transforms

$$X(w) = \int\limits_{c-i\infty}^{c+i\infty} x(s)e^{sw}\,ds, \tag{26}$$

$$F(w) = \int\limits_{c-i\infty}^{c+i\infty} f(s)e^{sw}\,ds = \int\limits_{c-i\infty}^{c+i\infty} \frac{\pi \sin \pi s}{\cos \pi s(\cos \pi s-1)}e^{sw}\,ds, \tag{27}$$

where $b < c < 1$ and $-d < v = \operatorname{im} w < d$. Equation (24) is now transformed into

$$(e^{-w}-1)X(w) = F(w)$$

with the solution
$$X(w) = -\frac{e^w}{e^w-1}F(w). \tag{28}$$

The solution of (24) is now obtained by means of the inverse transformation

$$x(s) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} X(w)e^{-ws}\,dw = -\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{e^w}{e^w-1}F(w)e^{-ws}\,dw, \tag{29}$$

where $b < \operatorname{re} s < 1$. It may be observed that $w = 0$ is *not* a pole of (28) because

$$F(0) = \int\limits_{c-i\infty}^{c+i\infty} \frac{d}{ds}\left[\log\frac{\cos \pi s}{\cos \pi s-1}\right]ds = 0.$$

In fact, this is the reason why we introduced the factor $(\sin \frac{1}{2}\pi s)^{-1}$ in (22).

The general solution of (24) is now obtained by adding to (29) any function with period 1. However, such a function cannot satisfy the simultaneous requirements that it tends to zero for $|s| \to \infty$ in the strip $b < \operatorname{re} s < 1$ and that it is regular in the strip $b < \operatorname{re} s < 2$, and it must therefore be omitted.

The evaluation of $F(w)$ (27) and $x(s)$ (29) is performed by standard methods of the calculus of residues. The results are

$$F(w) = -2\pi i\,\frac{e^{\frac{3}{2}w}(e^{\frac{1}{2}w}-1)}{(e^{\frac{1}{2}w}+1)(e^w+1)}, \tag{30}$$

† The author is indebted to his collaborator Mr. J. B. Alblas for the suggestion that a homogeneous difference equation may take a more convenient form by applying logarithmic differentiation.

in agreement with $F(0) = 0$, and

$$x(s) = \frac{\pi}{2\sin\pi s}+\frac{\pi(3-2s)}{\sin 2\pi s}. \tag{31}$$

It is easily verified that $x(s)$ is regular in the strip $\frac{1}{2} < \operatorname{re} s < 2\frac{1}{2}$ and that it satisfies the assumptions made with the introduction of its Laplace transform. A solution of (21) with the 'boundary' condition $T_0(1) = 1$ is now given by

$$T_0(s) = \frac{2^{s-1}\Gamma(s)}{\sin\frac{1}{2}\pi s}\exp\left[\int\limits_1^s \left\{\frac{\pi}{2\sin\pi z}+\frac{\pi(3-2z)}{\sin 2\pi z}\right\} dz\right], \tag{32}$$

where $\frac{1}{2} < \operatorname{re} s < 2$ and the line of integration lies in this strip. The exponential factor is bounded for $|s| \to \infty$ in this strip and its cofactor tends to zero exponentially. Our result therefore satisfies all requirements on $T_0(s)$.

It should be noted that the solution of (21) with 'boundary' condition $T_0(1) = 1$ is not unique. In fact, alternative solutions are obtained by multiplying (32) by any function $Q(s)$ of period 1 which is regular in the strip $b < \operatorname{re} s < 2$ and equal to unity for $s = 1$. However, such a function is at least of order $e^{2\pi|s|}$ for $|s| \to \infty$ in this strip, and the alternative solutions therefore violate the requirement $T_0(s) \to 0$ for $|s| \to \infty$ in this strip. Hence (32) is the *only* solution which satisfies *all* requirements on $T_0(s)$.

The exponential factor in (32) may be expressed in Alexeiewsky's G-function, defined by (**5**, p. 264),

$$G(z+1) = (2\pi)^{\frac{1}{2}z}e^{-\frac{1}{2}z(z+1)-\frac{1}{2}\gamma z^2}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n e^{-z+z^2/2n}\right\}, \tag{33}$$

where γ is again the Euler–Mascheroni constant. We write

$$\begin{aligned}
&\int\limits_1^s \left\{\frac{\pi}{2\sin\pi z}+\frac{\pi(3-2z)}{\sin 2\pi z}\right\} dz \\
&= \frac{1}{2}\int\limits_1^s \left\{\frac{\pi}{\sin\pi z}+\pi(3-2z)(\cot\pi z+\tan\pi z)\right\} dz \\
&= \frac{1}{2}\int\limits_1^s \left\{\frac{\pi}{\sin\pi z}-2\pi(z-1)\cot\pi z+\pi\cot\pi z+2\pi(z-\tfrac{3}{2})\cot\pi(z-\tfrac{1}{2})\right\} dz \\
&= \frac{1}{2}\int\limits_1^s \left\{\frac{\pi(1+\cos\pi z)}{\sin\pi z}-2\pi(z-1)\cot\pi(z-1)+2\pi(z-\tfrac{3}{2})\cot\pi(z-\tfrac{3}{2})\right\} dz \\
&= \int\limits_1^s \tfrac{1}{2}\pi\frac{\cos\frac{1}{2}\pi z}{\sin\frac{1}{2}\pi z}\, dz-\int\limits_0^{s-1}\pi z'\cot\pi z'\, dz'+\int\limits_{-\frac{1}{2}}^{s-\frac{3}{2}}\pi z''\cot\pi z''\, dz'',
\end{aligned}$$

where in the second integral we have written $z-1 = z'$ and in the third integral $z-\frac{3}{2} = z''$. Applying the formula (**5**, p. 264)

$$\int_0^s \pi z \cot \pi z\, dz = \log \frac{G(1-s)}{G(1+s)} + s \log 2\pi, \tag{34}$$

we obtain after a simple reduction

$$\int_1^s \left\{\frac{\pi}{2\sin \pi z} + \frac{\pi(3-2z)}{\sin 2\pi z}\right\} dz = \log\left\{\sin \tfrac{1}{2}\pi s \, \frac{G(\frac{1}{2})G(s)G(\frac{5}{2}-s)}{G(\frac{3}{2})G(2-s)G(-\frac{1}{2}+s)}\right\}. \tag{35}$$

The result (35) remains valid for all s in the complex s-plane except when the line of integration in the left-hand member passes through or ends in one of the poles of $x(s)$ defined in (31), viz. $s = 0$ or $s = 1 \pm n$ $(n = 2, 3, 4,...)$ or $s = \frac{3}{2} \pm k$ $(k = 1, 2, 3,...)$. Using the recurrence relation (**5**, p. 264)

$$G(z+1) = \Gamma(z)G(z), \tag{36}$$

expression (32) may now be simplified into

$$T_0(s) = \frac{2^{s-1}}{\Gamma(\frac{1}{2})} \frac{G(s+1)G(\frac{5}{2}-s)}{G(s-\frac{1}{2})G(2-s)}. \tag{37}$$

It is now easily verified that (37) indeed satisfies (21) in virtue of (36) and the gamma function formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

It may be observed that $T_0(s)$ is a *meromorphic* function because $G(z+1)$ is an *integral* function with n-tuple zeros $z = -n$ $(n = 1, 2, 3,...)$. The poles of $T_0(s)$ are n-tuple poles $s = n+1$ $(n = 1, 2, 3,...)$ and k-tuple poles $s = \frac{3}{2}-k$ $(k = 1, 2, 3,...)$.

The required non-dimensional shear stress distribution between the stiffener and the sheet is now obtained by means of the inverse Mellin transform

$$\tau_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} T_0(s) x^{-s}\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-1}}{\Gamma(\frac{1}{2})} \frac{G(s+1)G(\frac{5}{2}-s)}{G(s-\frac{1}{2})G(2-s)} x^{-s}\, ds, \tag{38}$$

where $\frac{1}{2} < c < 2$. The axial load $P(x)$ in the stiffener is given by

$$\frac{P(x)}{P_0} = 1 - \int_0^x \tau_0(\xi)\, d\xi = 1 - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{T_0(s)}{1-s} x^{1-s}\, ds$$

$$= 1 - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-1}}{\Gamma(\frac{1}{2})(1-s)} \frac{G(s+1)G(\frac{5}{2}-s)}{G(s-\frac{1}{2})G(2-s)} x^{1-s}\, ds, \tag{39}$$

where $\frac{1}{2} < c < 1$ and x is the non-dimensional coordinate.

5. Evaluation of solution

Evaluation of (38) and (39) by numerical integration is cumbersome. However, it will be shown that the line of integration may be closed in the left half-plane $\operatorname{re} s < c$ if the contour does not pass through any pole of (37). We take a rectangular contour with sides $\operatorname{re} s = c$, $\operatorname{re} s = -k+\frac{3}{4}$, where k is a large integer, and $\operatorname{im} s = \pm\tau$, where $\tau \to \infty$. It is easily seen from the bounds on the exponential factor in (32) that the contributions of the sides $\operatorname{im} s = \pm\tau$ tend to zero for $\tau \to \infty$. Furthermore, we write

$$\begin{aligned} T_0(s) &= \frac{2^{s-1}\Gamma(s)}{\Gamma(\frac{1}{2})}\,\frac{G(s)G(\frac{5}{2}-s)}{G(2-s)G(s-\frac{1}{2})} \\ &= \frac{2^{s-1}\Gamma(s)}{\Gamma(\frac{1}{2})}\exp\left\{-\int\limits_0^{s-1}\pi z\cot\pi z\,dz+(s-1)\log 2\pi+\right. \\ &\qquad\qquad \left.+\int\limits_0^{s-\frac{3}{2}}\pi z\cot\pi z\,dz-(s-\tfrac{3}{2})\log 2\pi\right\} \\ &= \frac{2^{s-1}\Gamma(s)}{\Gamma(\frac{1}{2})}\exp\left\{-\int\limits_{s-\frac{3}{2}}^{s-1}\pi z\cot\pi z\,dz+\tfrac{1}{2}\log 2\pi\right\}, \end{aligned} \tag{40}$$

where the exponential factor is seen to be $O(e^k)$ on the line $\operatorname{re} s = -k+\frac{3}{4}$. It is now obvious from the asymptotic expansion of the gamma function that the contribution of the side $\operatorname{re} s = -k+\frac{3}{4}$ of the rectangular contour tends to zero for $k \to \infty$.

The solution may now be written in the form of the series

$$\tau_0(x) = \sum_{k=1}^{\infty} \text{residues of } [T_0(s)x^{-s}] \text{ in } s = -k+\tfrac{3}{2}, \tag{41}$$

$$\frac{P(x)}{P_0} = 1-\sum_{k=1}^{\infty} \text{residues of } \left[\frac{T_0(s)}{1-s}x^{1-s}\right] \text{ in } s = -k+\tfrac{3}{2}, \tag{42}$$

where $T_0(s)$ is given by (37), and the series are convergent for all $x > 0$. Because $s = -k+\frac{3}{2}$ is a k-tuple pole, these series may be written in the form

$$\tau_0(x) = \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left\{\frac{d^{k-1}}{ds^{k-1}}[(s+k-\tfrac{3}{2})^k T_0(s)x^{-s}]\right\}_{s=-k+\frac{3}{2}}, \tag{43}$$

$$\frac{P(x)}{P_0} = 1-\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left\{\frac{d^{k-1}}{ds^{k-1}}\left[(s+k-\tfrac{3}{2})^k\frac{T_0(s)}{1-s}x^{1-s}\right]\right\}_{s=-k+\frac{3}{2}}. \tag{44}$$

The first few terms of the series (43) and (44) may be calculated in a straightforward manner, using the formulae

$$\frac{d}{ds}f(s) = f(s)\frac{d}{ds}\log f(s),$$

$$\frac{d^2}{ds^2}f(s) = f(s)\left[\frac{d^2}{ds^2}\log f(s)+\left\{\frac{d}{ds}\log f(s)\right\}^2\right], \quad \text{etc.}$$

By means of the recurrence relation (36) the final result for the stiffener load is obtained after some elementary but tedious algebra:

$$\frac{P(x)}{P_0} = 1-\sqrt{\left(\frac{2x}{\pi}\right)}+\frac{x}{3\pi}\sqrt{\left(\frac{2x}{\pi}\right)}[\tfrac{5}{3}+\log 2+\psi(\tfrac{3}{2})-\log x]-$$

$$-\frac{x^2}{30\pi^2}\sqrt{\left(\frac{2x}{\pi}\right)}[-\tfrac{1}{2}\pi^2+\tfrac{4}{25}-\psi'(\tfrac{5}{2})+\{\tfrac{7}{5}+\log 2+\psi(\tfrac{5}{2})-\log x\}^2]+...$$

$$= 1-\sqrt{\left(\frac{2x}{\pi}\right)}[1-x(0\cdot 25425-0\cdot 10610\log x)+$$

$$+x^2\{0\cdot 0086265-0\cdot 018888\log x+0\cdot 0033774(\log x)^2\}+...], \tag{45}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, and dashes denote differentiations.

The series in (45) is rapidly convergent when x is not too large, say $x < 2$, as may be seen from Fig. 4, where the curves a, b, c show the result by taking consecutively 1, 2, or 3 terms of the series between brackets in (45). It should be noted that between x around 0·5 and x around 1·5 the curve c lies below curve b, the maximum difference being approximately 2 per cent., and that from a value of x slightly above 1·5 curve c lies above curve b, the difference becoming important for $x > 2$. For large x the convergence is slow and the series (43) and (44) are not convenient for numerical computations. In fact, it appears that for x between, say, 2·5 and 5 the approximation by three terms between brackets in (45), curve c, shows even larger errors than the approximation by two terms, curve b.

However, a useful approximation for large x is obtained from the asymptotic expansion, deduced from (39) by contour integration in the half-plane re $s > c$. The first few terms of this expansion are readily written down as the residues in the simple poles $s = 1$ and $s = 2$ and in the double pole $s = 3$. The resulting asymptotic approximation for large x is

$$\frac{P(x)}{P_0} = \frac{2}{\pi x}-\frac{4}{\pi^2 x^2}(\log 2-\gamma-\log x)+.... \tag{46}$$

This result, retaining one or two terms in (46), is also depicted in Fig. 4 (curves d and e), and it is obvious that a satisfactory result for all $x > 0$ is obtained by linking the convergent expansion for small and moderate x with the asymptotic expansion for large x. This link has been drawn in Fig. 4 (curve f); the maximum possible error is estimated to be at most a few per cent.

A comparison with Buell's results shows very satisfactory agreement. Buell's curve for the stiffener load agrees with curve f in Fig. 4 within a few per cent. Our solution may serve to judge whether the number of unknown coefficients $b_n/\bar{P}$ (notation of (**1**)) retained by Buell is adequate.

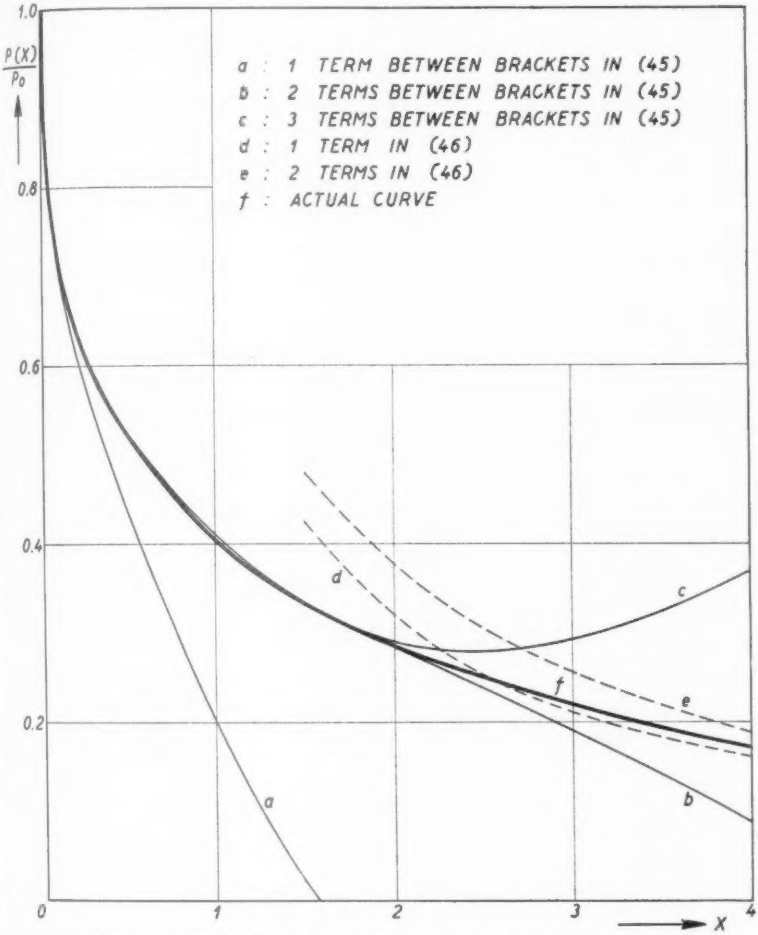

FIG. 4. The stiffener load as a function of the non-dimensional coordinate.

According to his solution the first two terms corresponding with our expansion (45) are (**1**, equation (42))

$$\frac{P(x)}{P_0} = 1 - \sqrt{x}\left[\frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\pi b_n}{\overline{P}}\right]. \tag{47}$$

The equivalence of (47) with the first two terms in (45) requires

$$\sum_{n=1}^{\infty}\frac{n\pi b_n}{\overline{P}} = \sqrt{(\tfrac{1}{2}\pi)} - 1 = 0{\cdot}253314. \tag{48}$$

Buell has solved his infinite system of equations approximately by retaining at most six equations and the first six coefficients. His result yields

$$\sum_{n=1}^{6} \frac{n\pi b_n}{P} = 0\cdot 247108,$$

indicating an error in the verification of (48) of around 2·5 per cent.

Finally it may be remarked that the numerical work required for evaluation of the present rigorous solution is negligible in contrast to Buell's approximate solution. However, this reduction in numerical computations has been bought at the expense of considerably more analytical work, involving more advanced methods.

Note added 8 March 1955

A. Pflüger ('Halbscheibe mit Randglied', *Zeitschr. f. angew. Math. u. Mech.* 25/27 (1947), 177) has also observed, independently from Benscoter, that the basic integro-differential equation is formally identical with Prandtl's equation.

REFERENCES

1. E. L. Buell, 'On the distribution of plane stress in a semi-infinite plate with partially stiffened edge', *J. Math. and Phys.* **26** (1948), 223.
2. S. U. Benscoter, 'Analysis of a single stiffener on an infinite sheet', *J. Applied Mech.* **16** (1949), 242.
3. S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 2nd ed. (McGraw-Hill, 1951), art. 33.
4. E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937).
5. E. T. Whittaker and G. N. Watson, *Modern Analysis*, 4th ed. (Cambridge, 1927).

SOUND SCATTERING AND TRANSMISSION BY THIN ELASTIC RECTANGULAR PLATES†

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SUMMARY

Scattering and transmission of sound waves of somewhat more general form than simple plane waves are analysed for scatterers in the form of a thin elastic simply-supported rectangular plate in a rigid infinite baffle. The plate is assumed to separate two different fluid media, and the sound radiation from the plate undergoing forced vibrations under the influence of the incident wave pressure is studied. Unlike the solid scatterer, whose dynamic characteristics are little different in vacuum and in certain fluid media, the characteristics of elastic plates are profoundly altered by fluid reaction. By correlating the general solution of the wave equation in the fluid media with that of the Lagrange equations of the elastic plate at the plate–fluid media interfaces, the deflexion of the plate motion is determined. A transmission coefficient τ is then defined and derived. In the appendixes, expressions for an energy ratio η and the reactions at the plate supports are given.

1. Introduction

THE scattered and transmitted patterns which result from the incidence of a pressure wave on an elastic plate separating two fluid media may markedly differ from the patterns obtained from a rigid scatterer of the same form. This difference, recently analysed for the case of an elastic shell (**1**), is due to the fact that the alternating sound pressure of the incident wave excites forced vibrations of the plate. The nature of these vibrations is determined by the dynamic characteristics of the plate separating the two fluid media. These characteristics, distinct from those of a plate vibrating in vacuum, depend on the density, elastic constants, dimensions, and boundary conditions of the plate, and on the densities of and the sound velocities in the two adjacent fluid media. It is desired to find the sound pressures for the scattered and transmitted waves in terms of these characteristic parameters of the plate and the fluid media.

The following assumptions are made: (1) the fluid media satisfy the conditions for the validity of the simplified wave equation; (2) the incident wave pressure is of the form

$$p_1(x,y,z,t) = C_{123}\exp i(-a_1 x - a_2 y)\exp i\omega(t - z/c_1),$$

where ω, a_1, and a_2 are such that $\omega/c_1 \gg a_1$ and $\omega/c_1 \gg a_2$; (3) the plate is

† This paper draws on research carried out for the Bureau of Ships, Department of the Navy, U.S.A., by Reed Research, Inc.

composed of a material which is isotropic and devoid of damping, and which obeys Hooke's law; (4) the dynamic deflexions and the thickness of the plate are small compared to its other dimensions; and (5) the plate boundaries are simply supported.

The standard methods of the theory of mechanical vibrations are used in the analysis. The dynamic configuration of the plate is expressed in terms of an infinite series (Fourier) whose coefficients are the generalized coordinates of the system. The potential and kinetic energies of the deformed plate are then calculated. The sound pressure, in the form of the general solution of the wave equation in the fluid media, is integrated over the surface of the plate to obtain the generalized forces of the plate. The Lagrange equations for the system are then introduced. Another set of equations is obtained from the continuity conditions, equating the normal component of the plate mid-surface displacement to the normal component of the fluid particle displacements at the plate interfaces. Simultaneous solutions of these two sets of equations yield an expression for the generalized coordinates. With these a ratio between the incident and the transmitted rates of energy, called 'transmission coefficient', is determined. The series expression of this ratio is obtained in terms of densities and acoustical velocities of both fluid media, the plate panel dimensions, area density, and flexural rigidity. Derivations of another energy ratio η, and expressions for support reactions are also included.

2. The total excess pressure (upon the plate)

The pressure wave equation for an undamped system and no sources in the region under consideration is

$$c_1^2\,\nabla^2 p = \frac{\partial^2 p}{\partial t^2}, \tag{1}$$

where p is the sound pressure, t is the time, and c_1 is the velocity of sound in fluid medium 1.

In Cartesian coordinates, the complete solution of (1) is given (**2**) by

$$p = \sum_{a_x}\sum_{a_y}\sum_{a_z} C_{xyz}\exp i(\pm a_x x \pm a_y y \pm a_z z \pm \omega_n t), \qquad a_x^2+a_y^2+a_z^2 = \omega_n^2/c_1^2, \tag{2}$$

where for every set of values of a_x, a_y, and a_z there will be a discrete value of angular frequency which is designated ω_n in (2). If we assume simple harmonic motion of $e^{+i\omega t}$ type of dependence on time, only one set of values a_x, a_y, $a_z \equiv a_1$, a_2, a_3 remains, with one related frequency $\omega_n \equiv \omega$, so that we can write

$$p = C_{123}\exp i(\pm a_1 x \pm a_2 y \pm a_3 z \pm \omega t), \qquad a_1^2+a_2^2+a_3^2 = \omega^2/c_1^2. \tag{2a}$$

or in abbreviated form

$$p = p(x,y,z)\exp(i\omega t) = C_{123}\,X_p(x)Y_p(y)Z_p(z)\exp(i\omega t). \tag{2 b}$$

Newton's equation of motion gives a relation between the particle velocity vector $\dot{\mathbf{u}}$ and the pressure,

$$\rho\frac{\partial\dot{\mathbf{u}}}{\partial t} = -\operatorname{grad} p, \tag{3}$$

where ρ is the fluid density.

The particle displacement vector corresponding to the solution for a simple harmonic wave of the type (2 b) is

$$\mathbf{u} = \mathbf{u}_0(x,y,z)\exp(i\omega t), \tag{4}$$

and by using (3) the velocity vector may be written as

$$\dot{\mathbf{u}} = +\frac{1}{i\omega}\frac{\partial\dot{\mathbf{u}}}{\partial t} = -\frac{1}{i\omega\rho}\operatorname{grad} p. \tag{5}$$

For cases where $\omega/c_1 \gg a_1$ and $\omega/c_1 \gg a_2$, $\dot{u}_{x_1}$ and $\dot{u}_{y_1}$ are small compared with $\dot{u}_{z_1}$ and may be neglected. We are then only interested in the z-component of $\dot{\mathbf{u}}$, which for fluid medium 1 is

$$\dot{u}_{z_1} = \frac{X_{p_1}(x)Y_{p_1}(y)}{-i\omega\rho_1}\,\frac{\partial Z_{p_1}(z)}{\partial z}\exp(i\omega t). \tag{6}$$

On the interface between medium 1 and the plate $z = -\frac{1}{2}h \doteq 0$ we can write for the particular velocity

$$(\dot{u}_{z_1})_{z=0} = \frac{X_{p_1}(x)Y_{p_1}(y)}{-i\omega\rho_1}\exp(i\omega t)\left(\frac{\partial Z_{p_1}(z)}{\partial z}\right)_{z=0}, \tag{7}$$

or, using (2 b),

$$(\dot{u}_{z_1})_{z=0} = \frac{p_1(x,y,0)}{-i\omega\rho_1}\left[\frac{\partial Z_{p_1}(z)}{\partial z}\Big/ Z_{p_1}(z)\right]_{z=0}\exp(i\omega t). \tag{8}$$

We limit now the nature of the scattered wave pressure in medium 1 to a simple harmonic form

$$p_1' = p_1'(x,y,z)\exp(i\omega t) = X'_{p_1}(x)Y'_{p_1}(y)Z'_{p_1}(z)\exp(i\omega t), \tag{9}$$

which according to (5) produces a velocity along the z-axis equal to

$$\dot{u}'_{z_1} = \frac{X'_{p_1}(x)Y'_{p_1}(y)}{-i\omega\rho_1}\,\frac{\partial Z'_{p_1}(z)}{\partial z}\exp(i\omega t), \tag{10}$$

where $\dot{u}'_{x_1}$ and $\dot{u}'_{y_1}$, which are small for small plate deflexions, are assumed to be negligible compared with $\dot{u}'_{z_1}$.

At the interface $z = -\frac{1}{2}h \doteq 0$ we have

$$(\dot{u}'_{z_1})_{z=0} = \frac{p'_1(x,y,0)}{-i\omega\rho_1}\left[\frac{\partial Z'_{p_1}(z)}{\partial z}\Big/ Z'_{p_1}(x)\right]_{z=0} \exp(i\omega t). \tag{11}$$

For the transmitted wave pressure in fluid medium 2 we again confine its nature to a simple harmonic form

$$p_2 = p_2(x,y,z)\exp(+i\omega t) = X_{p_2}(x)Y_{p_2}(y)Z_{p_2}(z)\exp(i\omega t), \tag{12}$$

which according to (5) renders a velocity along the z-axis

$$\dot{u}_{z_2} = \frac{X_{p_2}(x)Y_{p_2}(y)}{-i\omega\rho_2}\frac{\partial Z_{p_2}(z)}{\partial z}\exp(i\omega t). \tag{13}$$

Here $\dot{u}_{x_2}$ and $\dot{u}_{y_2}$ are also assumed negligible compared with $\dot{u}_{z_2}$. At the interface $z = +\frac{1}{2}h \doteq 0$ we get

$$(\dot{u}_{z_2})_{z=0} = \frac{p_2(x,y,0)}{-i\omega\rho_2}\left[\frac{\partial Z_{p_2}(x)}{\partial z}\Big/ Z_{p_2}(z)\right]_{z=0} \exp(i\omega t). \tag{14}$$

The condition of continuity requires at the plate interfaces

$$(\dot{u}_{z_1})_{z=0}+(\dot{u}'_{z_1})_{z=0} = \dot{w} = (\dot{u}_{z_2})_{z=0} \tag{15}$$

or

$$\left.\begin{aligned}(\dot{u}'_{z_1})_{z=0} &= \dot{w}-(\dot{u}_{z_1})_{z=0}\\ (\dot{u}_{z_2})_{z=0} &= \dot{w}\end{aligned}\right\}, \tag{16}$$

where w is the normal displacement of the plate.

The total excess normal pressure on the plate is

$$\begin{aligned}[p(x,y,z,t)]_{z=0} &= p(x,y,0,t)\\ &= p_1(x,y,0)\exp(i\omega t)+p'_1(x,y,0)\exp(i\omega t)-p_2(x,y,0)\exp(i\omega t).\end{aligned} \tag{17}$$

Using (8), (11), and (14), however, (17) reduces to the form

$$\begin{aligned}p(x,y,0,t) = {}& -i\omega\rho_1\left[Z_{p_1}(z)\Big/\frac{\partial Z_{p_1}(z)}{\partial z}\right]_{z=0}(\dot{u}_{z_1})_{z=0}-\\ &-i\omega\rho_1\left[Z'_{p_1}(z)\Big/\frac{\partial Z'_{p_1}(z)}{\partial z}\right]_{z=0}(\dot{u}_{z_1})_{z=0}+i\omega\rho_1\left[Z_{p_2}(z)\Big/\frac{\partial Z_{p_2}(z)}{\partial z}\right]_{z=0}(\dot{u}_{z_2})_{z=0},\end{aligned} \tag{18}$$

and using (16) we obtain

$$\begin{aligned}p(x,y,0,t) = {}& -i\omega\rho_1\left[Z_{p_1}(z)\Big/\frac{\partial Z_{p_1}(z)}{\partial z}-Z'_{p_1}(z)\Big/\frac{\partial Z'_{p_1}(z)}{\partial z}\right]_{z=0}(\dot{u}_{z_1})_{z=0}-\\ &-i\omega\left[\rho_1\, Z'_{p_1}(z)\Big/\frac{\partial Z_{p_1}(z)}{\partial z}-\rho_2\, Z_{p_2}(z)\Big/\frac{\partial Z_{p_2}(z)}{\partial z}\right]_{z=0}\dot{w}.\end{aligned} \tag{19}$$

Using (2 a) we find

$$\left.\begin{aligned} Z_{p_1}(z) &= \exp\left\{-iz\sqrt{\left(\frac{\omega^2}{c_1^2}-a_1^2-a_2^2\right)}\right\} = \exp\left(-i\frac{\omega}{\bar{c}_1}z\right) \\ Z'_{p_1}(z) &= \exp\left\{+iz\sqrt{\left(\frac{\omega^2}{c_1^2}-a_1^2-a_2^2\right)}\right\} = \exp\left(+i\frac{\omega}{\bar{c}_1}z\right) \\ Z_{p_2}(z) &= \exp\left\{-iz\sqrt{\left(\frac{\omega^2}{c_2^2}-a_1^2-a_2^2\right)}\right\} = \exp\left(-i\frac{\omega}{\bar{c}_2}z\right) \end{aligned}\right\}, \tag{20}$$

where

$$\left.\begin{aligned} \bar{c}_1 &= \left(\frac{1}{c_1^2}-\frac{a_1^2}{\omega^2}-\frac{a_2^2}{\omega^2}\right)^{-\frac{1}{2}} \\ \bar{c}_2 &= \left(\frac{1}{c_2^2}-\frac{a_1^2}{\omega^2}-\frac{a_2^2}{\omega^2}\right)^{-\frac{1}{2}} \end{aligned}\right\}. \tag{20 a}$$

and

We then have

$$\left.\begin{aligned} \frac{\partial Z_{p_1}(z)}{\partial z} &= -i\frac{\omega}{\bar{c}_1}Z_{p_1}(z) \\ \frac{\partial Z'_{p_1}(z)}{\partial z} &= +i\frac{\omega}{\bar{c}_1}Z'_{p_1}(z) \\ \frac{\partial Z_{p_2}(z)}{\partial z} &= -i\frac{\omega}{\bar{c}_2}Z_{p_2}(z) \end{aligned}\right\}, \tag{20 b}$$

which when substituted in (19) give finally the total excess pressure in the form

$$p(x,y,0,t) = 2\rho_1\bar{c}_1(\dot{u}_{z_1})_{z=0}-(\rho_1\bar{c}_1+\rho_2\bar{c}_2)\dot{w}, \tag{21}$$

or, using (8) and (20),

$$p(x,y,0,t) = 2p_1(x,y,0)\exp(i\omega t)-(\rho_1\bar{c}_1+\rho_2\bar{c}_2)\dot{w}. \tag{22}$$

It can be noted that, under the assumptions in this analysis, we always have $(\omega/c)^2 \gg a_1^2$ and $(\omega/c)^2 \gg a_2^2$ in both fluid media, which results in $\bar{c}_1 \doteq c_1$ and $\bar{c}_2 \doteq c_2$, so that (22) may be approximately written as follows:

$$p(x,y,0,t) = 2p_1(x,y,0)\exp(i\omega t)-(\rho_1 c_1+\rho_2 c_2)\dot{w}. \tag{23}$$

3. The Lagrange equations and their solutions

Consider the plane $z = 0$ at the middle surface of the undeflected flat plate of thickness h with its edges at $x = 0$, $y = 0$, $x = a$, and $y = b$. For small deflexions of a thin plate the effect of the static deflexions upon the dynamic deflexions may be neglected. The equation of motion for the transverse dynamic deflexions of the plate is

$$D\nabla^4 w+m\ddot{w} = p(x,y,0,t), \tag{24}$$

where D is the flexural rigidity of the plate.

For a simply supported plate, $w(x,y,t)$ can be expressed in the form of a double Fourier series

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{25}$$

where the coefficients $q_{mn}(t)$ are to be determined as solutions of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_{mn}}\right) - \frac{\partial T}{\partial q_{mn}} + \frac{\partial V}{\partial q_{mn}} = Q_{mn} \tag{26}$$

and are the generalized coordinates. In (26) the Q_{mn} are the generalized forces. The potential energy or strain energy V for the plate is

$$V = \frac{D}{2}\int_0^a\int_0^b \left\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right\}^2 dxdy. \tag{27}$$

Substituting (25) into (27) we have

$$V = \frac{ab\bar{m}}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \omega_{mn}^2 q_{mn}^2, \tag{28}$$

where ω_{mn} is the natural frequency of the mnth mode of the plate vibration. The kinetic energy T of the vibrating plate is given by

$$T = \frac{\bar{m}}{2}\int_0^a\int_0^b \dot{w}^2\, dxdy, \tag{29}$$

which on using (25) becomes

$$T = \frac{ab\bar{m}}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \dot{q}_{mn}^2, \tag{30}$$

where $\bar{m}$ is the surface density of the plate.

The mnth generalized force may be written as

$$Q_{mn} = \int_0^a\int_0^b p(x,y,0,t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dxdy. \tag{31}$$

Substituting (23) and (25) into (31), we have

$$Q_{mn} = 2\exp(i\omega t)\int_0^a\int_0^b p_1(x,y,0)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dxdy - \\ -(\rho_1 c_1 + \rho_2 c_2)\frac{ab}{4}\dot{q}_{mn}. \tag{32}$$

Expressing now $p_1(x,y,0)$ by a double Fourier series of the form

$$p_1(x,y,0) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \lambda_{rs}\sin\frac{r\pi x}{a}\sin\frac{s\pi y}{b}, \tag{33}$$

we get
$$\int_0^a\int_0^b p_1(x,y,0)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy = \frac{ab}{4}\lambda_{mn}. \tag{34}$$

Hence (32) gives
$$Q_{mn} = \frac{ab}{2}\lambda_{mn}\exp(i\omega t)-(\rho_1 c_1+\rho_2 c_2)\frac{ab}{4}\dot{q}_{mn}. \tag{35}$$

Substituting (35), (28), and (30) into (26), we finally obtain the Lagrange equations in terms of the generalized coordinates
$$\ddot{q}_{mn}+\frac{\rho_1 c_1+\rho_2 c_2}{\bar{m}}\dot{q}_{mn}+\frac{D\pi^4}{\bar{m}}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)q_{mn} = \frac{2\lambda_{mn}}{\bar{m}}\exp(i\omega t). \tag{36}$$

It can be seen that the transient solutions will die out and the steady-state solutions of (36) are as follows:
$$q_{mn} = \frac{\exp(i\omega t)}{i\omega Z_{mn}}, \tag{37}$$
where Z_{mn}, which is of the nature of a mechanical impedance, is given by
$$Z_{mn} = \frac{\bar{m}}{2i\omega}\left[(\omega_{mn}^2-\omega^2)+\frac{i\omega}{\bar{m}}(\rho_1 c_1+\rho_2 c_2)\right]$$
$$= \tfrac{1}{2}(\rho_1 c_1+\rho_2 c_2)-i\frac{\bar{m}\omega}{2}(x_{mn}^2-1). \tag{38}$$

Corresponding to the imaginary part of p_1, i.e.
$$\operatorname{im} p_1 = C_{123}\cos(a_1 x+a_2 y)\sin\omega t-C_{123}\sin(a_1 x+a_2 y)\cos\omega t,$$
(37) may be written as
$$q_{mn} = -|q_{mn}|\cos(\omega t+\phi_{mn}+\alpha_{mn}), \tag{39}$$
where
$$|q_{mn}| = \frac{2|\lambda_{mn}|}{\omega(\rho_1 c_1+\rho_2 c_2)}\left[1+\beta_{mn}^2\left(\frac{x_{mn}^2-1}{x_{mn}}\right)^2\right]^{-\frac{1}{2}}, \tag{39 a}$$
$$\phi_{mn} = \tan^{-1}\frac{\beta_{mn}(x_{mn}^2-1)}{x_{mn}}, \tag{39 b}$$
and
$$\alpha_{mn} = \tan^{-1}\frac{\operatorname{im}\lambda_{mn}}{\operatorname{re}\lambda_{mn}}, \tag{39 c}$$
in which
$$x_{mn} = \frac{\omega_{mn}}{\omega}; \qquad \beta_{mn} = \frac{\bar{m}}{\rho_1 c_1+\rho_2 c_2}\omega_{mn};$$
$$\omega_{mn} = \pi\left(\frac{D}{\bar{m}}\right)^{\frac{1}{4}}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right). \tag{40}$$

4. The transmission coefficient τ

We define, after Vogel (3), the transmission coefficient τ as the ratio of mean rate of energy transmitted to the second fluid medium at the inner

interface 2 over mean rate of incident energy in the first medium arriving at the outer interface 1:

$$\tau = \frac{\overline{W}_2^{T_0}}{\overline{W}_1^{T_0}}. \tag{41}$$

According to reference (**3**), the mean rate of vibratory energy of the incident wave arriving at the interface over a period T_0 can be expressed (under the assumption that $\dot{u}_{x_1}$ and $\dot{u}_{y_1}$ are negligible compared with $\dot{u}_{z_1}$) as

$$\overline{W}_1^{T_0} = \frac{1}{T_0}\int\limits_0^{T_0} dt \int\limits_0^a \int\limits_0^b \tfrac{1}{2}\,\mathrm{im}[p_1(x,y,0,t)]\,\mathrm{im}[\dot{u}_{z_1}(x,y,0,t)]\,dxdy. \tag{42}$$

From (33) we have

$$\mathrm{im}[p_1(x,y,0,t)] = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} |\lambda_{rs}|\sin\frac{r\pi x}{a}\sin\frac{s\pi y}{b}\sin(\alpha_{rs}+\omega t), \tag{43}$$

and from (8) and (20 b)

$$\mathrm{im}[\dot{u}_{z_1}(x,y,0,t)] = \frac{1}{\rho_1 c_1}\mathrm{im}[p_1(x,y,0,t)]. \tag{44}$$

Substituting (33), (43), and (44) in (42) and simplifying, we get

$$\overline{W}_1^{T_0} = \frac{ab}{16\rho_1 c_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\lambda_{mn}|^2. \tag{45}$$

Similarly the mean rate of transmitted energy at the interface 2 over a period T_0 is

$$\overline{W}_2^{T_0} = \frac{1}{T_0}\int\limits_0^{T_0} dt \int\limits_0^a \int\limits_0^b \tfrac{1}{2}\,\mathrm{im}[p_2(x,y,0,t)]\,\mathrm{im}[\dot{u}_{z_2}(x,y,0,t)]\,dxdy. \tag{46}$$

Substituting (14) and (20) into (46), we obtain

$$\overline{W}_2^{T_0} = \frac{1}{T_0}\int\limits_0^{T_0} dt \int\limits_0^a \int\limits_0^b \tfrac{1}{2}\rho_2 c_2\,\dot{w}^2(x,y,t)\,dxdy. \tag{47}$$

By virtue of (25) and (39) and carrying out the integrations,

$$\overline{W}_2^{T_0} = \frac{ab}{16}\rho_2 c_2\,\omega^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |q_{mn}|^2. \tag{47 a}$$

Substituting (45) and (47 a) into (41) gives

$$\tau = \frac{\overline{W}_2^{T_0}}{\overline{W}_1^{T_0}} = \frac{(\rho_1 c_1)(\rho_2 c_2)\omega^2}{\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}|\lambda_{mn}|^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |q_{mn}|^2. \tag{48}$$

Using the expression (39 a) for $|q_{mn}|$, we finally obtain

$$\tau = \frac{4(\rho_1 c_1)(\rho_2 c_2)}{(\rho_1 c_1+\rho_2 c_2)^2}\,\frac{1}{\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}|\lambda_{mn}|^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{|\lambda_{mn}|^2}{1+\beta_{mn}^2\{(x_{mn}^2-1)/x_{mn}\}^2}, \tag{49}$$

where β_{mn} and x_{mn} are defined in (40) and λ_{mn} is given (see (34)) by

$$\lambda_{mn} = \operatorname{re}\lambda_{mn} + i\operatorname{im}\lambda_{mn}, \tag{50}$$

in which

$$\operatorname{re}\lambda_{mn} = \frac{4mn\pi^2 C_{123}}{[(m\pi)^2-(a_1 a)^2][(n\pi)^2-(a_2 b)^2]}\times$$
$$\times[(-1)^{m+n}\cos(a_1 a+a_2 b)-(-1)^m\cos a_1 a-(-1)^n\cos a_2 b+1], \tag{50 a}$$

$$\operatorname{im}\lambda_{mn} = \frac{4mn\pi^2 C_{123}}{[(m\pi)^2-(a_1 a)^2][(n\pi)^2-(a_2 b)^2]}\times$$
$$\times[(-1)^{m+n}\sin(a_1 a+a_2 b)-(-1)^m\sin a_1 a-(-1)^n\sin a_2 b]. \tag{50 b}$$

5. Example on evaluation of τ

Let us consider a plane incident wave $p_1(x,y,0,t) = p_0\sin\omega t$. We find, from (50),

$$|\lambda_{mn}| = \lambda_{mn} = \frac{4p_0}{ab}\int\limits_0^a\int\limits_0^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy = \begin{cases} 16p_0/\pi^2 mn & \text{for } m, n \text{ odd,} \\ 0 & \text{for other combinations of } m \text{ and } n. \end{cases} \tag{51}$$

Noting that

$$\sum_m^{1,3,5\ldots}\sum_n^{1,3,5\ldots}\frac{1}{m^2n^2} = \sum_m^{1,3,5\ldots}\frac{1}{m^2}\sum_n^{1,3,5\ldots}\frac{1}{n^2} = \sum_m^{1,3,5\ldots}\frac{1}{m^2}\left(\frac{\pi^2}{8}\right) = \left(\frac{\pi^2}{8}\right)\left(\frac{\pi^2}{8}\right) = \frac{\pi^4}{64}, \tag{52}$$

we obtain from (49) and (51)

$$\tau = \frac{64\times 4}{\pi^4}\,\frac{(\rho_1 c_1)(\rho_2 c_2)}{(\rho_1 c_1+\rho_2 c_2)^2}\sum_m^{1,3,5\ldots}\sum_n^{1,3,5\ldots}\frac{1}{m^2n^2[1+\beta_{mn}^2\{(x_{mn}^2-1)/x_{mn}\}^2]}. \tag{53}$$

If the two media are the same, $\rho_1 c_1 = \rho_2 c_2$, and if we use the approximation $\pi^2 \doteq 10$, we get

$$\tau = \sum_m^{1,3,5\ldots}\sum_n^{1,3,5\ldots}\frac{0{\cdot}64}{m^2n^2}\,\frac{1}{[1+\beta_{mn}^2\{(x_{mn}^2-1)/x_{mn}\}^2]}, \tag{54}$$

which is the result Vogel obtained in reference (3) (equation (19)).

Vogel further showed that, for a medium of low density (e.g. air) and for a source of high frequency (e.g. sound source), (54) can be approximated by a finite inequality as follows:

$$\frac{0{\cdot}32}{ab\omega}K < \tau < \frac{0{\cdot}32}{ab\omega}K+\frac{4(\rho_1 c_1)^2}{\bar{m}^2\omega^2}; \qquad K = \pi^2\sqrt{\frac{D}{\bar{m}}}. \tag{55}$$

By similar reasoning it may be shown that, for the case of low density and high frequency, (53) can be approximated as follows:

$$\left(\frac{128}{\pi^2}\right)\frac{(\rho_1 c_1)(\rho_2 c_2)}{(\rho_1 c_1+\rho_2 c_2)^2}\frac{1}{ab\omega}\sqrt{\frac{D}{\bar{m}}} < \tau$$
$$< \left(\frac{128}{\pi^2}\right)\frac{(\rho_1 c_1)(\rho_2 c_2)}{(\rho_1 c_1+\rho_2 c_2)^2}\frac{1}{ab\omega}\sqrt{\frac{D}{\bar{m}}}+\frac{4(\rho_1 c_1)(\rho_2 c_2)}{\bar{m}^2\omega^2}. \quad (56)$$

6. Energy ratio and reaction forces

If energy of the vibrating plate rather than the transmitted energy is of interest, a new coefficient called 'energy ratio' η may be defined as

$$\eta = \frac{\text{mean elastic energy absorbed by the plate}}{\text{mean incident energy in the first medium at the interface 1}}.$$

This ratio is derived in Appendix A. It is an infinite series and may also be expressed in terms of τ.

When the dynamic deflexion of the plate is determined, the periodic reaction forces at the edge supports R_x, R_y can be easily evaluated from well-known formulae in the theory of plates. They are derived in Appendix B.

While the transmission coefficient τ is of interest from the point of view of acoustics, both energy ratio η and reaction forces R_x and R_y are of more immediate concern from the point of view of elasticity.

7. Conclusions

The analysis gives expressions for the plate deformation w and the transmission coefficient τ in terms of various parameters of the plate and the two adjacent fluid media. In Appendixes A and B the expressions for the energy ratio η and the support reactions R_x and R_y are derived.

The results apply to cases where $\omega/c \gg a_1$ and $\omega/c \gg a_2$. This includes problems where: (1) the velocities of sound in the fluid media are low and the frequency of the incident wave is high, i.e. the wavelengths are so small that they may be neglected as compared with $2\pi/a_1$ and $2\pi/a_2$; and (2) the incident, reflected, and transmitted waves are so close to a plane wave that $a_1 \doteq 0$ and $a_2 \doteq 0$ in both media.

It is interesting to note in extending the work of Vogel to the more general case of two different fluid media adjacent to the plate and to a somewhat more general form of the incident wave that such quantities as τ, η, R_x, and R_y are still representable in relatively simple algebraic forms.

The authors believe that the present analysis can be extended to plates with other boundary conditions, as well as stiffened plates and plates with structural damping. Further investigation with still more realistic incident, reflected, and transmitted wave forms will be rewarding.

APPENDIX A

The Energy Ratio η

We define a new coefficient η called the energy ratio as follows:

$$\eta = \frac{\text{mean elastic energy absorbed by the plate}}{\text{mean incident energy in the first medium at the interface 1}} = \frac{E_0^{T_0}}{E_1^{T_0}}. \tag{A1}$$

The mean incident energy at the interface 1 over one period can be obtained by integrating the mean rate of incident energy given in (42)

$$E_1^{T_0} = \int_0^t \overline{W}_1^{T_0}\,dt = \frac{1}{T_0}\int_0^{T_0}\left(\int_0^t dt\right)dt \int_0^a\int_0^b \tfrac{1}{2}\,\mathrm{im}[p_1(x,y,0,t)]\,\mathrm{im}[\dot{u}_{z_1}(x,y,0,t)]\,dxdy. \tag{A2}$$

By virtue of (33), (43), and (44)

$$E_1^{T_0} = \frac{ab\pi}{16\omega\rho_1 c_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\lambda_{mn}|^2\left(1+\frac{1}{2\pi}\sin 2\alpha_{mn}\right) \doteq \frac{ab\pi}{16\omega\rho_1 c_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\lambda_{mn}|^2$$

$$(\text{for } \alpha_{mn} \doteq 0). \tag{A3}$$

In order to compute the mean elastic energy absorbed by the plate over one period, we define

$$E_0^{T_0} = \frac{1}{T_0}\int_0^{T_0} dt \int_0^t \tfrac{1}{2}\,\mathrm{im}[p(x,y,0,t)]\,dw \int_0^a\int_0^b dxdy, \tag{A4}$$

or we can write

$$E_0^{T_0} = \frac{1}{2T_0}\int_0^{T_0}\left(\int_0^t dt\right)dt \int_0^a\int_0^b \mathrm{im}[p(x,y,0,t)]\dot{w}(x,y,t)\,dxdy. \tag{A5}$$

Using (22), (33), (25), and (39) in (A5), we find

$$E_0^{T_0} = \frac{\pi ab}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\lambda_{mn}|\,|q_{mn}| - \frac{\omega ab\pi(\rho_1 c_1+\rho_2 c_2)}{16}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|q_{mn}|^2, \tag{A6}$$

and by means of (39a) we get

$$E_0^{T_0} = \frac{\pi ab}{4\omega(\rho_1 c_1+\rho_2 c_2)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\lambda_{mn}|^2}{1+\beta_{mn}^2\{(x_{mn}^2-1)/x_{mn}\}^2}\left\{\left[1+\beta_{mn}^2\left(\frac{x_{mn}^2-1}{x_{mn}}\right)^2\right]^{\frac{1}{2}}-1\right\}. \tag{A7}$$

Substitution of (A3) and (A7) into (A1) renders the expression of the energy ratio η

$$\eta = \frac{4\rho_1 c_1}{(\rho_1 c_1+\rho_2 c_2)}\,\frac{1}{\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}|\lambda_{mn}|^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\lambda_{mn}|^2}{1+\beta_{mn}^2\{(x_{mn}^2-1)/x_{mn}\}^2}\times$$

$$\times\left\{\left[1+\beta_{mn}^2\left(\frac{x_{mn}^2-1}{x_{mn}}\right)^2\right]^{\frac{1}{2}}-1\right\}. \tag{A8}$$

Comparing (A8) with (49), we may express η in terms of τ as

$$\eta = \frac{4\rho_1 c_1}{\rho_1 c_1+\rho_2 c_2}\,\frac{1}{\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}|\lambda_{mn}|^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\lambda_{mn}|^2}{[1+\beta_{mn}^2\{(x_{mn}^2-1)/x_{mn}\}^2]^{\frac{1}{2}}} - \frac{\rho_1 c_1+\rho_2 c_2}{\rho_2 c_2}\tau. \tag{A9}$$

APPENDIX B

Reaction Forces at the Plate Edges

The vertical reactions on the supports of a simply supported plate due to the dynamic loading (excess pressures only) can be expressed as

$$R_x = -D\left[\frac{\partial^3 w}{\partial x^3}+(2-\mu)\frac{\partial^3 w}{\partial x\partial y^2}\right] \quad \text{along } x=0 \quad \text{and} \quad x=a \tag{B1}$$

and

$$R_y = -D\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial y\partial x^2}\right] \quad \text{along } y=0 \quad \text{and} \quad y=b, \tag{B2}$$

where μ is the Poisson's ratio of the material of the plate.

By virtue of (25)

$$R_x = D\pi^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(\frac{m}{a}\right)^3+(2-\mu)\left(\frac{m}{a}\right)\left(\frac{n}{b}\right)^2\right]q_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{B3}$$

Hence, after substituting (39) into (B3),

$$\left.\begin{aligned}
R_x &= -D\pi^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m}{a}\right)\left[\left(\frac{m}{a}\right)^2+(2-\mu)\left(\frac{n}{b}\right)^2\right]\sin\frac{n\pi y}{b}\times\\
&\qquad\times|q_{mn}|\cos(\omega t+\phi_{mn}+\alpha_{mn}) \quad \text{at } x=0,\\
&\text{and}\\
R_x &= -D\pi^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^m\left(\frac{m}{a}\right)\left[\left(\frac{m}{a}\right)^2+(2-\mu)\left(\frac{n}{b}\right)^2\right]\sin\frac{n\pi y}{b}\times\\
&\qquad\times|q_{mn}|\cos(\omega t+\phi_{mn}+\alpha_{mn}) \quad \text{at } x=a;
\end{aligned}\right\} \tag{B4}$$

and similarly

$$\left.\begin{aligned}
R_y &= -D\pi^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n}{b}\right)\left[\left(\frac{n}{b}\right)^2+(2-\mu)\left(\frac{m}{a}\right)^2\right]\sin\frac{m\pi x}{a}\times\\
&\qquad\times|q_{mn}|\cos(\omega t+\phi_{mn}+\alpha_{mn}) \quad \text{at } y=0,\\
&\text{and}\\
R_y &= -D\pi^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{b}\right)\left[\left(\frac{n}{b}\right)^2+(2-\mu)\left(\frac{m}{a}\right)^2\right]\sin\frac{m\pi x}{a}\times\\
&\qquad\times|q_{mn}|\cos(\omega t+\phi_{mn}+\alpha_{mn}) \quad \text{at } y=b.
\end{aligned}\right\} \tag{B5}$$

The values of R_x and R_y so derived represent the periodic forces per unit edge length acting on the supports at edges $x = 0$, $x = a$ and $y = 0$, $y = b$ respectively, due to the excess pressures. Actual reactions can be obtained by superimposing these values on the reactions due to the static loading, if any.

REFERENCES

1. M. Junger, 'Sound scattering by thin elastic shells', *Journal of Acoust. Soc. of America*, July 1952, p. 366.
2. A. Bronwell, *Advanced Mathematics in Physics and Engineering* (McGraw-Hill, 1953), p. 241.
3. T. Vogel, *Sur les vibrations de certains systèmes élastiques dans un champ sonore*, Publications Scientifiques et Techniques du Ministère de l'Air, No. 209 (1948).

A BOUNDARY LAYER PHENOMENON IN THE LARGE DEFLEXION OF THIN PLATES

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SUMMARY

In the large deflexion of thin plates with free edges, boundary layers develop along those edges under certain conditions. The non-linear effects of membrane stress are confined to the boundary layers and elsewhere the equations are linear. Boundary-layer equations are derived and integrated, and are then applied to some specific problems.

1. Introduction

As long ago as 1912 H. Reissner (**1**) pointed out the existence of a boundary-layer phenomenon in the bending of thin shells. Suppose, for example, that the shell is subjected to a pressure on its surface together with tractions and bending moments on its edge. Then, if the thickness is small enough, the deformation can be determined approximately by neglecting the flexural rigidity of the shell and assuming that the applied loads are resisted entirely by the membrane stresses. This approximation breaks down, however, near the edge of the shell where, in a narrow region, or boundary layer, the deformation changes rapidly to satisfy the conditions on the edge.

A different type of edge effect was described (**2**) by Kelvin and Tait in 1867 in order to give a physical explanation to Kirchhoff's boundary conditions for an edge which is subjected to shear force, twisting moment, and bending moment. Using variational methods Kirchhoff had previously shown that the two conditions involving shear force and twisting moment combine to give only one condition. In explanation Kelvin and Tait pointed out that twisting moments of line intensity M_{sn} acting on the edge are statically equivalent to shear forces of intensity $\partial M_{sn}/\partial s$, where s is the distance along the edge. The combination of shear forces V_n and twisting moments M_{sn} on the edge are therefore statically equivalent to shear forces of intensity $(V_n+\partial M_{sn}/\partial s)$. On applying Saint-Venant's principle it follows that if a solution is obtained which gives the correct values for the bending moments and these equivalent shear forces on the edge, it will give an accurate description of the state of stress and deformation in the plate except in a very narrow region near the edge. This edge region, or boundary layer, was discussed by Friedrichs (**3**) in 1949. Friedrichs showed that the

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width of the boundary layer is of the order of the thickness of the plate, a fact which is of importance in the work to be described later.

The present paper is concerned with yet another type of boundary-layer phenomenon. This may best be illustrated by a consideration of an initially flat long strip, of width $2b$ and thickness t, bent uniformly in the longitudinal direction into an arc of a circle of radius R.

It is well known that simple bending theory when applied to such a strip involves an anticlastic curvature equal to $-\mu/R$ in the direction of the width of the strip, where μ is Poisson's ratio. Experiments show, however, that this anticlastic curvature does not necessarily occur; in fact with very thin plates the cross-section remains practically undistorted. This phenomenon was discussed by Searle (**4**) and by Case (**5**) who both concluded that the cross-section distortion depends upon the value of a parameter b^2/Rt. If $b^2/Rt \leqslant 1$ the anticlastic curvature corresponding to simple bending theory develops, but if b^2/Rt becomes large the cross-section remains almost undistorted. The transition stage between the small and large values of the parameter has recently been investigated by Ashwell, both theoretically (**6**, **8**) and experimentally (**7**). He showed that as b^2/Rt increases the distortion of the cross-section becomes more and more confined to a narrow region or boundary layer adjacent to each edge of the strip. The membrane stresses which exist in the strip are also confined to the boundary layers. Thus when the longitudinal curvature is large the strip assumes on the whole a cylindrical form, with the generators perpendicular to the two edges, and in a state of pure bending. Disturbances from this state occur in narrow boundary layers along the edges, where the middle surface, being no longer developable, becomes stretched and membrane stresses appear. The width of the boundary layers is of the order $\surd(Rt)$.

It is clear that throughout the cylindrical portion there must exist in the plate a uniform lateral bending moment equal to μ times the longitudinal bending moment, in order that the lateral curvature should be zero. This lateral bending moment is obviously zero on each edge and it therefore increases through the width of the boundary layer from zero at the edge to its required value inside. This is the important function of the boundary layer, and the mechanism by which it accomplishes it is as follows. The longitudinal membrane stresses which exist in the boundary layer, in combination with the longitudinal curvature, can be thought of as giving rise to radial forces. If then we consider a narrow lateral strip of the plate as a beam, it is subjected to normal forces distributed over the two end regions. The resultant of these forces is a pure bending moment about a longitudinal axis which corresponds exactly to the lateral bending moment required to prevent the lateral curvature.

The purpose of the present paper is to attempt to generalize this concept of the boundary layer to apply to certain problems involving large deflexions of plates in which a similar phenomenon occurs but which are not as amenable to a rigorous analysis as the simple case of a uniform strip under uniform bending discussed above.

A particularly interesting case is that of a horizontal flat square plate subjected to four equal forces at its corners, two of them upwards at the ends of one diagonal, and the other two downwards at the ends of the other diagonal. According to the linear, small deflexion theory the plate undergoes a uniform twist, so that any line parallel to an edge remains straight. It requires only the simplest of experiments to see that if the deflexions are appreciable the surface into which the plate deforms is quite different from that given by linear theory. It will be found, in fact, that the surface appears to be perfectly cylindrical, the generators being parallel to one or other of the two diagonals. The plate can be forced from one to the other of these two equally possible modes, although once it has started to deform in one mode it will continue to do so under the action of the corner forces alone. (The reader can easily verify this for himself by means of a square piece of card of the type used for filing purposes.) Such a cylindrical mode of deformation does not satisfy the boundary conditions associated with linear theory. It can be made to do so, however, by introducing a boundary layer, of the type discussed above, along each edge of the plate. In this case the generators of the cylindrical surface are inclined to the edge and in order to deal with the situation the von Kármán equations for large deflexion are used, in conjunction with an order argument, to set up boundary-layer equations which apply in this more general case.

The problem of the torsion of a square plate described above is a particular case of the torsion of a rhombic plate by forces applied at the corners. In this case the deformation mode is again cylindrical, although it will be found that the mode which develops is always that which has generators parallel to the shorter diagonal. If the deformation is large enough, however, it can be forced into the other mode, in which the generators are parallel to the longer diagonal. This second mode appears to be stable only if the curvature is larger than a certain critical value; below this value it becomes unstable and changes suddenly to the first mode. The solution of this problem, which again involves a boundary layer, is given in the paper. In order to deal with the stability problem involved, however, a considerably more complete solution would be required.

The other solutions which are given are concerned with the large deflexions of cantilevered plates which are loaded so that they deform into a cylindrical mode.

Since the work described in this paper was completed two interesting facts have been brought to the authors' notice. First, the transition between the small and large deflexion modes of a twisted square plate was apparently first noticed by Kelvin and Tait (**9**), who describe it as a 'remarkable case ... which deserves particular notice; not only as interesting in itself and important in practical application, but as curiously illustrating one of the most difficult points in the general theory'. Secondly, the general concept of thin plates deforming into developable surfaces has been used in two papers by Mansfield and Kleeman (**10**, **11**) who are concerned mainly with the deflexion of swept and triangular cantilever plates. The work described in the present paper is somewhat more fundamental than that of Mansfield and Kleeman in that it explains how a developable surface can be made to fit the boundary conditions of the problem, a difficulty which was not considered by Mansfield and Kleeman.

2. Uniform bending of a long strip

Consider an initially flat long strip, of width $2b$ and uniform thickness t, under uniform bending as shown in Fig. 1. If the longitudinal curvature is $1/R$, the simple theory of bending predicts that the cross-section assumes a lateral curvature equal to $-\mu/R$. It is shown by Ashwell (**6**) that this is a true picture only if a parameter αb, defined by

$$\alpha b = [3(1-\mu^2)]^{\frac{1}{4}} \frac{b}{(Rt)^{\frac{1}{2}}}, \tag{1}$$

is small, i.e. $\alpha b \ll 1$. If this parameter is large, however, the cross-section remains substantially flat except in a region adjacent to each edge of the strip, as shown in Fig. 2. The deviation ζ of the middle line of the cross-section from the axis through the centroid of the distorted section is given approximately by

$$\zeta = -\frac{\mu t}{\sqrt{\{12(1-\mu^2)\}}} e^{-\alpha y}(\cos \alpha y - \sin \alpha y) \tag{2}$$

in each of these two edge regions, or boundary layers, where y is the distance measured from the edge in question.

If $\mu = 0{\cdot}32$, the value of ζ at each edge is given by $\zeta_{y=0} = -0{\cdot}0975t$ whilst the first two values of y at which $\zeta = 0$ are equal to $0{\cdot}613\sqrt{(Rt)}$ and $3{\cdot}066\sqrt{(Rt)}$. Thus the width† δ of the boundary layer is of order $\sqrt{(Rt)}$ and, for large values, the parameter αb can be interpreted as expressing the ratio

† Just as in the flow of a viscous fluid over the surface of a body the width δ of the boundary layer is not a clearly defined quantity. We could define it specifically by stipulating that if $y > \delta$, $\zeta < \epsilon t$ where ϵ is some chosen small quantity. However, this is not necessary, and for our purposes it is the order and not the absolute magnitude which is of importance.

of the width of the plate to the width of the boundary layer. The horizontal scale to which Fig. 2 has been drawn corresponds to a value $\alpha b = 10$, which gives some idea of the value of αb required to produce such a boundary layer.

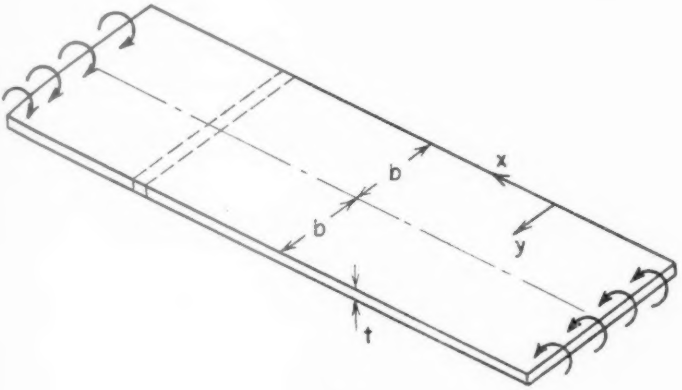

FIG. 1

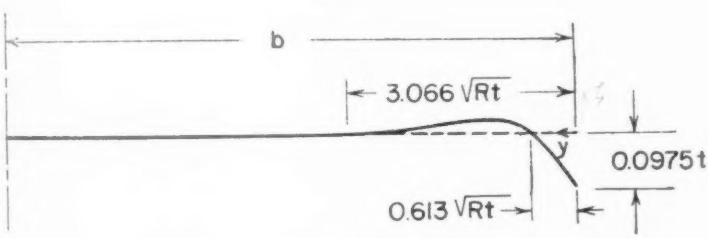

FIG. 2

It should be noticed that the effect of increasing the longitudinal curvature, once the boundary layer has formed, is to narrow down the width of the layer without changing the peak values of ζ. Thus the maximum deviation from flatness is only about 10 per cent. of the thickness, which in practice is difficult to detect. Thus if an experiment of this nature is performed, the surface appears to be truly cylindrical.

The membrane tension N_x in the longitudinal direction per unit width in the y direction is given by

$$N_x = -\frac{Et}{R}\zeta, \tag{3}$$

where E is Young's modulus. Thus the membrane tension is proportional to the distance of the middle line of the cross-section from the axis through

its centroid. Furthermore, the membrane tension is confined to the boundary layer and the middle region of the strip is in a state of pure bending.

It is clear that over the cylindrical middle region of the strip there must exist a lateral bending moment M_y equal to $-\mu D/R$ where D is the flexural rigidity of the plate defined by $D = Et^3/12(1-\mu^2)$. But M_y is zero on each edge of the strip. The mechanism by which this change in the value of M_y in passing through the boundary layer is accomplished is as follows: The membrane tension N_x in the boundary layer, combined with the longitudinal curvature $1/R$, provides an effective force N_x/R per unit area, in a direction perpendicular to the surface of the strip. Thus if we consider a lateral 'beam' of unit width (as shown dotted in Fig. 1) the membrane tensions produce a lateral bending moment M_y at a distance y from the edge given by

$$M_y = \int\limits_0^y \frac{1}{R} N_x(Y)(y-Y)\,dY.$$

For the purposes of this integration the cylindrical region between the two boundary layers corresponds to $y = \infty$ so that, on using equations (2) and (3), the total lateral bending moment produced by the boundary layer is given by

$$M_y = \frac{E\mu t^2}{R^2\sqrt{\{12(1-\mu^2)\}}} \int\limits_0^\infty (y-Y)e^{-\alpha Y}(\cos\alpha Y - \sin\alpha Y)\,dY.$$

On integrating and using equation (1) this becomes

$$M_y = -\frac{\mu D}{R} \tag{4}$$

which agrees with the required value. *Thus the important function of the boundary layer is to provide a lateral bending moment in the cylindrical middle region sufficient to inhibit lateral curvature.*

3. The boundary-layer equations

Suppose that in a plate undergoing large deflexions a boundary layer of the type being considered has developed along a free edge of the plate as shown in Fig. 3. Let the direction of the generators of the cylindrical surface be inclined at an angle $(\frac{1}{2}\pi-\theta)$ to the edge. Choose two sets of coordinate axes Ox, Oy and OY, OX; the first pair are parallel and perpendicular to the edge and the second pair parallel and perpendicular to the generators respectively.

We denote by w^* the deflexion of the cylindrical portion of the plate so that

$$\left.\begin{aligned} \frac{\partial^2 w^*}{\partial Y^2} &= \frac{\partial^2 w^*}{\partial X \partial Y} = 0 \\ \frac{\partial^2 w^*}{\partial X^2} &= \frac{1}{R_X} \end{aligned}\right\}, \tag{5}$$

where R_X is the radius of curvature of the cylinder. In general R_X is a function of X only.

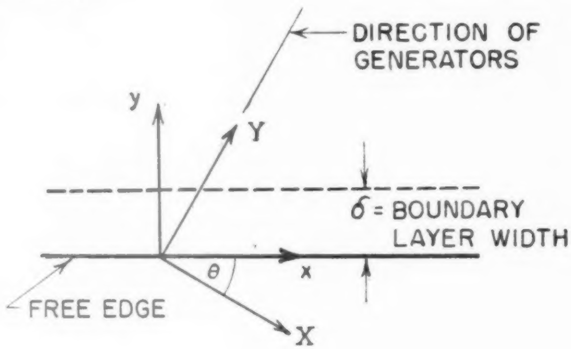

Fig. 3

On transforming from the X, Y to the x, y coordinates by means of the equations

$$\left.\begin{aligned} X &= x\cos\theta - y\sin\theta \\ Y &= x\sin\theta + y\cos\theta \end{aligned}\right\}$$

we find that

$$\frac{\partial^2 w^*}{\partial x^2} = \frac{\cos^2\theta}{R_X}, \qquad \frac{\partial^2 w^*}{\partial y^2} = \frac{\sin^2\theta}{R_X}, \qquad \frac{\partial^2 w^*}{\partial x \partial y} = -\frac{\sin\theta\cos\theta}{R_X}. \tag{6}$$

By hypothesis the tensions and shear force N_x, N_y, and N_{xy} per unit width are zero in the cylindrical region. In the boundary layer they satisfy the equations of equilibrium

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0; \qquad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0. \tag{7}$$

We now define l as a length characteristic of the length of the edge of the plate and δ as the width of the boundary layer. Also let

$$\xi = \frac{x}{l}, \qquad \eta = \frac{y}{\delta}, \tag{8}$$

so that in the boundary layer both ξ and η are $O(1)$. Equations (7) now become

$$\frac{\partial N_{xy}}{\partial \eta} = -\frac{\delta}{l}\frac{\partial N_x}{\partial \xi}; \qquad \frac{\partial N_y}{\partial \eta} = -\frac{\delta}{l}\frac{\partial N_{xy}}{\partial \xi}.$$

It will be seen therefore that

$$\frac{N_{xy}}{N_x} = O\left(\frac{\delta}{l}\right); \qquad \frac{N_y}{N_x} = O\left(\frac{\delta^2}{l^2}\right). \tag{9}$$

Again by hypothesis the width δ of the boundary layer is small compared with the typical dimension of the edge l, so that in the boundary layer we can neglect N_y and N_{xy} compared with N_x. Thus the membrane forces in the boundary layer consist primarily of tensions in the direction of the edge.

Now let the deflexion of the plate in the boundary layer be given by

$$w = w^* + \zeta, \tag{10}$$

where ζ represents the departure from the cylindrical form. The equation of equilibrium (**12**) of the plate in a direction perpendicular to the xy-plane is

$$D\nabla^4 w = q + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2}, \tag{11}$$

where q is the applied pressure on the surface, ∇^4 is the biharmonic operator, and D is the flexural rigidity of the plate defined as

$$D = \frac{Et^3}{12(1-\mu^2)}.$$

In the cylindrical portion, where the membrane stresses are zero, equation (11) reduces to

$$D\nabla^4 w^* = q. \tag{12}$$

Hence, on using equations (5), we see that q is a function of X only, which means that a cylindrical surface is possible only if the surface pressure is constant along each generator.

It will be seen from equations (6) that in general the values of $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, and $\partial^2 w/\partial x \partial y$ are of the same order of magnitude. Hence, on substituting equations (10) and (12) into equation (11) and neglecting the terms involving N_{xy} and N_y we obtain the equation

$$D\nabla^4 \zeta = N_x\frac{\partial^2 w}{\partial x^2}. \tag{13}$$

From equation (8) we have

$$\nabla^4 \zeta = \frac{1}{\delta^4}\frac{\partial^4 \zeta}{\partial \eta^4} + \frac{2}{\delta^2 l^2}\frac{\partial^4 \zeta}{\partial \xi^2 \partial \eta^2} + \frac{1}{l^4}\frac{\partial^4 \zeta}{\partial \xi^4},$$

and since $\delta/l \ll 1$, we need include only the first term of this expression, so that approximately

$$\nabla^4 \zeta = \frac{\partial^4 \zeta}{\partial y^4}.$$

Hence on using the first of equations (6), equation (13) becomes

$$\frac{\partial^4\zeta}{\partial y^4} = \frac{\cos^2\theta}{DR_X} N_x. \tag{14}$$

The second of the large deflexion equations of von Kármán (**12**) can be written in the form

$$\nabla^2(N_x+N_y) = Et\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]. \tag{15}$$

Now N_y can be neglected compared with N_x. Furthermore, we have

$$\nabla^2 N_x = \frac{1}{\delta^2}\frac{\partial^2 N_x}{\partial\eta^2} + \frac{1}{l^2}\frac{\partial^2 N_x}{\partial\xi^2},$$

and only the first term of this expression need be retained. Thus the left-hand side of equation (15) is approximately equal to $\partial^2 N_x/\partial y^2$.

Using equations (6) and (10) the term in square brackets on the right-hand side of equation (15) is equal to

$$-\left[\frac{2\sin\theta\cos\theta}{R_X}\frac{\partial^2\zeta}{\partial x\partial y} + \frac{\sin^2\theta}{R_X}\frac{\partial^2\zeta}{\partial x^2} + \frac{\cos^2\theta}{R_X}\frac{\partial^2\zeta}{\partial y^2}\right] + \left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right].$$

But
$$\frac{\partial^2\zeta}{\partial x\partial y}\bigg/\frac{\partial^2\zeta}{\partial y^2} = O\left(\frac{\delta}{l}\right) \tag{16}$$

and
$$\frac{\partial^2\zeta}{\partial x^2}\bigg/\frac{\partial^2\zeta}{\partial y^2} = O\left(\frac{\delta^2}{l^2}\right) \tag{17}$$

so that the first two terms in the above expression may be neglected in comparison with the third. Hence equation (15) now reduces to

$$\frac{1}{Et}\frac{\partial^2 N_x}{\partial y^2} = -\frac{\cos^2\theta}{R_X}\frac{\partial^2\zeta}{\partial y^2} + \left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]. \tag{18}$$

In the case of the uniform strip under pure bending discussed earlier we found that $\zeta = O(t)$ and that the width of the boundary layer $\delta = O(\sqrt{R_X t})$. We shall now assume that these orders still apply and confirm the assumption later. The ratio of the term in square brackets to the first term on the right-hand side of equation (18) is then of order δ^2/l^2 and hence only the first term need be considered. Equation (18) then reduces to

$$\frac{\partial^2 N_x}{\partial y^2} = -\frac{Et\cos^2\theta}{R_X}\frac{\partial^2\zeta}{\partial y^2}. \tag{19}$$

Equations (14) and (19) are the boundary-layer equations which take the place of von Kármán's large deflexion equations. If we multiply equations (14) and (19) we obtain the equation

$$\frac{\partial^4\zeta}{\partial y^4}\frac{\partial^2 N_x}{\partial y^2} = -\frac{12(1-\mu^2)\cos^4\theta}{t^2 R_X^2} N_x \frac{\partial^2\zeta}{\partial y^2}.$$

Hence on using the second of equations (8) we see immediately that

$$\delta = O(\sqrt{(R_X t)}). \tag{20}$$

Furthermore, it will be seen from the discussion of the uniform strip under pure bending that the membrane stresses in the boundary layer are responsible for developing a bending moment M_y in the cylindrical portion of the plate which is of order D/R_X. Hence

$$N_x \frac{1}{R_X}\delta^2 = O\Big(\frac{D}{R_X}\Big),$$

so that

$$N_x = O\Big(\frac{D}{\delta^2}\Big),$$

that is

$$N_x = O\Big(\frac{Et^2}{R_X}\Big). \tag{21}$$

Finally, from equation (17) we have

$$\zeta = O\Big(\frac{N_x R_X}{Et}\Big),$$

and on using equation (21) this becomes

$$\zeta = O(t). \tag{22}$$

Equations (20) and (22) confirm the assumption which was made in deriving equation (19).

We can integrate equations (19) by observing that if the boundary layer is thin we may assume that R_X is independent of y for the purposes of this integration. Thus

$$N_x = -\frac{Et\cos^2\theta}{R_X}\zeta + yf_1(x) + f_2(x),$$

where f_1 and f_2 are arbitrary functions of integration. But as y increases indefinitely both N_x and ζ approach zero so that $f_1 = f_2 = 0$. Hence we have

$$N_x = -\frac{Et\cos^2\theta}{R_X}\zeta, \tag{23}$$

so that the membrane stress in the boundary layer is proportional to ζ, the deviation of the middle surface from the cylindrical form.

On substituting equation (23) into equation (14) we now obtain the equation

$$\frac{\partial^4\zeta}{\partial y^4} + 4\alpha^4\zeta = 0, \tag{24}$$

where

$$\alpha^2 = [3(1-\mu^2)]^{\frac{1}{2}}\frac{\cos^2\theta}{t|R_X|}. \tag{25}$$

Again, assuming that in the boundary layer R_X is independent of y equation (24) can be integrated to give

$$\zeta = e^{-\alpha y}[A(x)\sin\alpha y + B(x)\cos\alpha y] + e^{\alpha y}[P(x)\sin\alpha y + Q(x)\cos\alpha y]. \quad (26)$$

But as y increases indefinitely ζ tends to zero so that $P = Q = 0$. In order to determine A and B we must now consider the conditions on the free edge. These conditions are that

$$\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } y = 0, \quad (27)$$

$$\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad \text{at } y = 0. \quad (28)$$

The first of these two equations expresses the fact that the bending moment M_y is zero on the edge; the second equation is the Kirchhoff boundary condition which says that the equivalent shear force is zero on the edge.

On using equations (6), (10), and (17) these two conditions become

$$\frac{\partial^2 \zeta}{\partial y^2} = -(\sin^2\theta + \mu\cos^2\theta)\frac{1}{R_X} \quad \text{at } y = 0,$$

$$\frac{\partial^3 \zeta}{\partial y^3} = \sin\theta[\sin^2\theta + (2-\mu)\cos^2\theta]\frac{d}{dX}\left(\frac{1}{R_X}\right) \quad \text{at } y = 0.$$

Consider the second of these two equations. On using equations (20) and (22) the left-hand side is of order $1/(R_X\,\delta)$ whilst the right-hand side is of order $1/(R_X\,l)$. Hence we may replace the right-hand side by zero since it is of a smaller order of magnitude than the left-hand side. The two conditions for determining A and B now become

$$\frac{\partial^2 \zeta}{\partial y^2} = -\frac{(\sin^2\theta + \mu\cos^2\theta)}{R_X}; \qquad \frac{\partial^3 \zeta}{\partial y^3} = 0, \quad \text{at } y = 0. \quad (29)$$

On substituting equation (26) we finally obtain

$$A = -B = \frac{t(\mu + \tan^2\theta)}{\sqrt{\{12(1-\mu^2)\}}},$$

and the expression for ζ becomes

$$\zeta = -\frac{t(\mu + \tan^2\theta)}{\sqrt{\{12(1-\mu^2)\}}}e^{-\alpha y}(\cos\alpha y - \sin\alpha y). \quad (30)$$

If the generators of the cylindrical surface are perpendicular to the free edge ($\theta = 0$) equation (30) becomes identical with equation (2) which defines the boundary layer of a uniformly bent strip. There is one essential difference, however; even if $\theta = 0$ the radius of curvature R_X of the cylindrical surface, and therefore α, is in general a function of x, whereas in the uniformly bent strip both R_X and α are constant.

4. Bending of cantilevered plates

Consider a thin plate $ABCD$ of the shape shown in Fig. 4, fixed along the edge AD and perfectly free otherwise. Suppose that it is subjected to a normal pressure on its surface which is a function of X only, so that along any line such as PQ the pressure is constant. In addition it may be subjected to a shear force and a bending moment M_x uniformly distributed over the

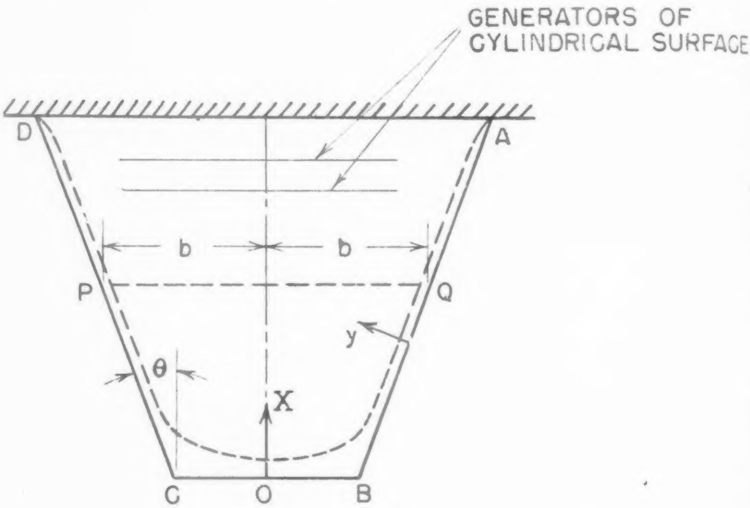

FIG. 4

tip section BC. Then, if the deflexion is large the plate will bend into a cylindrical surface, whose generators are parallel to the root AD, except for a boundary layer along each of the two edges AB and CD. The curvature of the cylinder can easily be calculated. For, if we neglect the contribution of the membrane stresses in the boundary layer to the total bending moment at a section such as PQ, we have

$$\frac{1}{R_X} = \frac{M}{2bD}, \tag{31}$$

where M is the total bending moment at the section PQ due to the applied loads and $2b$ is the width of the section.

The deviation of the plate from this cylindrical surface in the boundary layers is given by equation (30) where α is defined by equation (25) and R_X by equation (31). The membrane stress exists only in the boundary layers and is given by equation (23).

Clearly this is the solution only if the width of the boundary layer at any section is much less than the width of the plate, i.e. if $\sqrt{(tR_X)} \ll b$. It may be that the loading is such that it is impossible to satisfy this condition at all sections; for example in the case of uniform tip shear force the bending moment is zero at the tip, and however large the applied shear force, there will always be a region near the tip in which the condition for a boundary layer to form is not satisfied. The plate will then assume a cylindrical surface only inside a line such as that shown dotted in Fig. 4.

Furthermore, there can be no boundary layer at the root section AD since the clamp will ensure that there is no distortion of that section. Hence a small region near the root section must also be excluded.

The special case of a rectangular cantilevered plate is given by $\theta = 0$, and that of a triangular cantilevered plate is obtained by making B and C coincide. These are both included in the more general solution of the plate shown in Fig. 4.

It will be seen from equation (30) that the value of ζ at each free edge of the plate ($y = 0$) is given by

$$\zeta_0 = -\frac{t(\mu+\tan^2\theta)}{\sqrt{\{12(1-\mu^2)\}}}. \tag{32}$$

This is independent of R_X so that once the boundary layer has formed ζ_0 remains constant, both along the edge and with further increase of load. It will also be seen that ζ_0 increases numerically as θ increases. If $\mu = 0{\cdot}32$, for example, the numerical value of ζ_0 when $\theta = 45^\circ$ is over four times as great as when $\theta = 0$. Even then, however, it is only about 40 per cent. of the plate thickness so that the maximum deviation from the cylindrical form is not large.

5. Torsion of a rhombic plate

Suppose that a plate in the shape of a rhombus (Fig. 5) is loaded by four equal forces P at the corners, in a direction perpendicular to the plane of the plate. The forces at A and C are applied in the opposite direction to those at B and D. On the assumption that the deflexions are small enough for the linear small deflexion theory to apply, this problem has been solved by E. Reissner (**13**). The expression for the deflexion w of the plate obtained from Reissner's work is

$$w = \frac{P}{4D\sin 2\theta}\left[\frac{x_0^2-y_0^2}{1-\mu}+\frac{(x_0^2+y_0^2)\cos 2\theta}{1+\mu}\right], \tag{33}$$

where the x_0 and y_0 coordinates are as shown in Fig. 5. On differentiating this expression to obtain the curvatures of the surface we find that

$$\frac{\partial^2 w}{\partial x_0^2} = \frac{P}{2D\sin 2\theta}\left[\frac{1}{1-\mu}+\frac{\cos 2\theta}{1+\mu}\right],$$

$$\frac{\partial^2 w}{\partial y_0^2} = -\frac{P}{2D\sin 2\theta}\left[\frac{1}{1-\mu}-\frac{\cos 2\theta}{1+\mu}\right].$$

Thus the curvature is constant in both the x_0 and y_0 directions. Further-

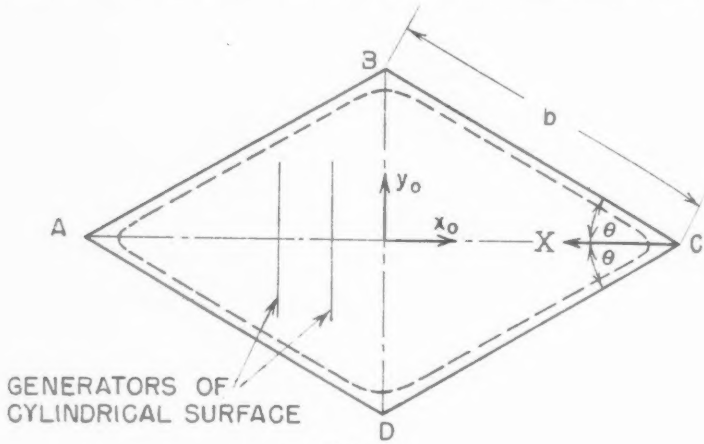

FIG. 5

more, the curvature in the x_0 direction is positive and that in the y_0 direction negative. Thus the plate bends into an anticlastic surface with constant principal curvatures.

Such a surface, however, is not developable and therefore the plate stretches, and membrane stresses are introduced. The presence of these membrane stresses modifies the deflexion surface and by the time the deflexions become large the shape bears no resemblance to that predicted by the linear theory. In fact a simple experiment showed that the plate bends into a cylindrical shape, the generators being parallel to the shorter diagonal of the plate. Such a deflexion mode does not satisfy the boundary conditions, however, but this can be explained by the presence of a boundary layer of the type under discussion which forms along each edge of the plate. This was easily visible in the experiment which was performed; also it was noticeable that near the corners, especially the acute ones, where the boundary layers interact there was an appreciable departure from the cylindrical shape. On the whole, however, the plate deflexions are very well represented by the cylindrical shape.

It is easy to calculate the curvature of the cylinder. If we again neglect the contribution of membrane stresses in the boundary layers to the total bending moment carried by a section parallel to BD at a distance X from C (where $X \leqslant b\cos\theta$) we have

$$PX = \frac{2X\tan\theta\, D}{R_X},$$

i.e.

$$\frac{1}{R_X} = \frac{P}{2D}\cot\theta. \tag{34}$$

Thus the curvature of the cylinder is constant, so that the width of the boundary layer is also constant. The deviation from the cylindrical form in the boundary layer is given by equation (30) and the membrane stresses by equation (23), where α is defined by equation (25) and R_X by equation (34).

Using either equation (33) or (34) it is a simple matter to calculate the relative deflexion of the corners A and C from the corners B and D from either the small deflexion theory or from the large deflexion theory given here. If these are denoted by Δ_s and Δ_l respectively we have

$$\Delta_s = \frac{Pb^2}{4(1-\mu^2)D}\sin 2\theta(1+\mu+2\cot^2 2\theta),$$

$$\Delta_l = \frac{Pb^2}{4D}\cot\theta\cos^2\theta.$$

Thus

$$\frac{\Delta_s}{\Delta_l} = \frac{2\tan^2\theta}{1-\mu^2}(1+\mu+2\cot^2 2\theta).$$

In the case of a square plate ($\theta = 45°$) with $\mu = 0{\cdot}32$ this gives $\Delta_s = 2{\cdot}94\Delta_l$, whilst if $\theta = 30°$, $\Delta_s = 1{\cdot}47\Delta_l$. Hence the small deflexion theory gives a considerable overestimate of the deflexion of the plate.

It is interesting to observe that a square plate can bend with the generators of the cylindrical surface parallel to either diagonal. Once it has started to deform in one mode it will continue to do so unless forced into the other mode. In the case of the rhombus, however, the plate always starts to deform in the mode with the generators parallel to the shorter diagonal. If the curvature is not very large and an attempt is made to force it into the other mode, with the generators parallel to the longer diagonal, it springs back into the first mode as soon as it is released. If the curvature is sufficiently large, however, the second mode appears to be stable though the plate will never deform into it without being forced.

6. Comparison with other well-known boundary layers

The novel character of the boundary layer discussed above can be brought out by a comparison with other well-known boundary-layer phenomena, such as Prandtl's boundary layer in a flow at large Reynolds number, or Kelvin, Tait, and Friedrich's boundary layer in the bending of plates with regard to Kirchhoff's boundary conditions. In these classical examples the effect of boundary layer on the field outside is such that the terms involving derivatives of the highest *order* in the complete differential equations are negligible. In contrast to this, the effect of the membrane stress boundary layer discussed above is to make the terms involving the highest *power* of the dependent variables negligible in the field outside the boundary layer. The equations of motion for the flow outside Prandtl's boundary layer remain non-linear, while the equations for the bending of the plate outside the membrane stress boundary layer become linearized. The highest derivatives in Prandtl's case represent the effect of viscosity of the fluid: when they are neglected the fluid behaves like a non-viscous one. The highest derivatives in the plate-bending equations represent the effect of bending rigidity: they are fully effective outside the membrane stress boundary layer.

It may be pointed out that although the derivation of the boundary layer presented in § 3 fulfils the purpose of the present paper, the basic assumptions can be made less restrictive to cope with more general loading conditions. For example, the assumption that the deflexion surface outside the boundary layer is cylindrical can be replaced by simply being developable; the tractions on the unsupported edges may be finite.

7. Some general observations regarding 'applicable surfaces' in the theory of plates and shells

From equation (15) it can be seen that if membrane stresses vanish in a plate the deflexion surface is a developable surface. This is related to a number of important phenomena concerning bending and buckling of thin shells. In differential geometry, two surfaces are said to be 'applicable' to each other if one can be deformed continuously into the other without stretching or tearing. A developable surface is applicable to a plane. Hence the phenomenon discussed in this paper can be stated as that under certain conditions the plate deflects in such a manner that, except for a boundary layer, it remains applicable to the original surface.

It is well known that when a cylindrical shell is subjected to a sufficiently large end compression it buckles into a diamond-shaped deflexion pattern (see sketch in Fig. 6). An ideal diamond pattern consists of a series of triangular flat surfaces and is applicable to the original cylinder. The actual

buckling pattern deviates slightly from the ideal applicable surface for the following reason. The bending rigidity of a shell varies as Et^3, while the rigidity resisting the stretching of the mid-surface of the shell varies as Et. Hence as the thickness t becomes very small, it becomes far easier to bend the shell than to stretch it. Hence it is natural to expect that the buckling mode of the shell should involve little stretching. However, if the buckled surface is exactly applicable to the original surface, the stretching of the mid-surface vanishes identically and there is no strain energy due to membrane stresses. The bending strain-energy, however, becomes large in spite of the small bending rigidity because of the high curvatures near the ridges. The actual physical phenomenon strikes a balance to require a small stretching strain energy to reduce the bending strain energy. The result is a mode that is *almost* applicable to the cylinder. Although it would be wrong to assume that the deflexion mode is exactly applicable to the cylinder, the recognition of the diamond pattern enables an approximate evaluation of the buckling load of cylinders to be obtained, as demonstrated by von Kármán and Tsien (**14**) and Leggett and Jones (**15**), etc. That the diamond pattern is almost applicable to the cylinder has been discussed recently by Yoshimaru (**16**).

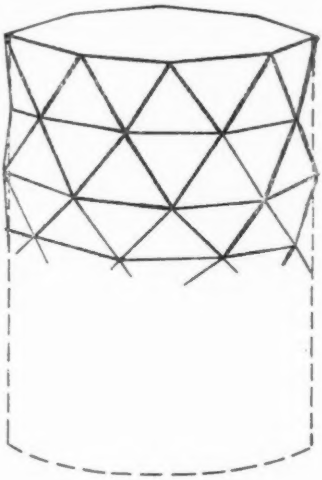

Fig. 6

Similar situations arise in a number of plate-bending problems. A case of interest to aeronautical engineers is that of the deformation of a swept wing under load. An idealization of this would be the case of a thin flat strip clamped obliquely at the root cross-section and subjected to some form of loading at the tip. A simple experiment with such a configuration appeared to indicate that the plate deformed into a surface which contained straight generators. These generators were perpendicular to the edges of the strip at some distance from the root, but in the root region they rotated until right at the support the generator coincided with the root section. The question therefore arises as to the possibility of obtaining a solution for this problem in which the mode of deformation is a developable surface with a boundary layer along each edge in order to deal with the edge conditions. If the loads are applied at the tip the deflexion function would

also have to be biharmonic in order to satisfy equilibrium requirements. However, it is shown in the appendix that it is impossible in general to satisfy the biharmonic equation and the developable condition simultaneously. The only surfaces which do so are cubic cylinders (and possibly

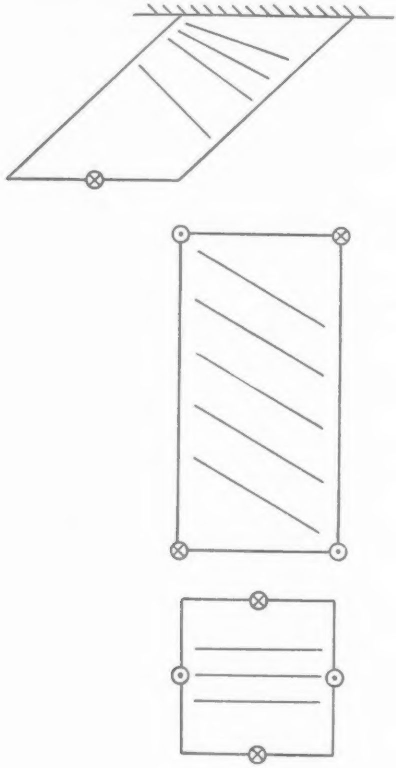

FIG. 7

conical surfaces although this has not been fully investigated). If the plate is loaded by means of a pressure on its surface the right-hand side of the biharmonic equation is no longer zero. However, in general, the distribution of pressure required to obtain a solution in the form of a developable surface is so unrealistic that further investigation is unwarranted. Thus the deflexion surface of a swept plate will not be an exact developable surface with a boundary layer along each edge. However, when the plate is very thin the deviation of the deflexion surface from a developable surface is small.

In Fig. 7 the lines of zero curvature for a swept plate under a load at the tip are sketched. Similar sketches are given for the deflexion pattern of a very thin rectangular plate which is twisted beyond the range of validity of the linear torsion theory, and that of the torsion of a rectangular plate subjected to forces not acting at the corners. These patterns can be demonstrated by very simple experiments. These deflexion patterns would be useful for deriving approximate solutions by energy methods.

APPENDIX

Surfaces that are both developable and biharmonic

With any developable surface other than a cylinder or a cone there is associated a line of regression. This is a space curve whose tangents sweep out the developable surface (**17**), (**18**).

Let $\mathbf{r}(s)$ be the position vector of a point on the line of regression, where s is a parameter specifying the position of the point on the line, i.e.

$$\mathbf{r}(s) = (f(s), g(s), h(s)).$$

Then the position vector of a point on the developable surface is

$$(x, y, w) = \mathbf{r}(s)+u\dot{\mathbf{r}}(s),$$

i.e.

$$\left.\begin{aligned} x &= f(s)+u\dot{f}(s) \\ y &= g(s)+u\dot{g}(s) \\ w &= h(s)+u\dot{h}(s) \end{aligned}\right\} \tag{35}$$

where u specifies the distance of the point along the tangent from the point of tangency to the line of regression, and a dot refers to differentiation with respect to s. Hence it can be shown that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\dot{g}^2}{u}\psi(s), \qquad \frac{\partial^2 w}{\partial y^2} = \frac{\dot{f}^2}{u}\psi(s), \tag{36}$$

where
$$\psi(s) = \frac{1}{(\ddot{f}\dot{g}-\dot{f}\ddot{g})^3}[\dot{h}(\dddot{f}\ddot{g}-\ddot{f}\dddot{g})+\ddot{h}(\dot{f}\dddot{g}-\dot{g}\dddot{f})+\dddot{h}(\ddot{f}\dot{g}-\dot{f}\ddot{g})].$$

For a one-to-one correspondence between the x, y and u, s coordinate systems, the denominator of the above expression must not vanish. Writing

$$\phi(s) = (\dot{f}^2+\dot{g}^2)\psi(s)$$

and
$$J(s) = \ddot{f}\dot{g}-\dot{f}\ddot{g}$$

it can be shown further that

$$\nabla^4 w = \frac{1}{J}\left\{\frac{1}{u^3}\left[\dot{f}\frac{d}{ds}\left(\frac{\dot{f}\dot{\phi}+\ddot{f}\phi}{J}\right)+\dot{g}\frac{d}{ds}\left(\frac{\dot{g}\dot{\phi}+\ddot{g}\phi}{J}\right)+\frac{2}{J}\{(\dot{f}\ddot{f}+\dot{g}\ddot{g})\dot{\phi}+(\ddot{f}^2+\ddot{g}^2)\phi\}\right]+\right.$$
$$\left.+\frac{1}{u^4}\left[\dot{f}\frac{d}{ds}\left(\frac{\dot{f}\phi}{J}\right)+\dot{g}\frac{d}{ds}\left(\frac{\dot{g}\phi}{J}\right)+\frac{2}{J}\{(\dot{f}\ddot{f}+\dot{g}\ddot{g})\phi+(\dot{f}^2+\dot{g}^2)\dot{\phi}\}+\frac{3\phi}{J}(\dot{f}\ddot{f}+\dot{g}\ddot{g})\right]+\frac{3}{u^5}\frac{\phi}{J}(\dot{f}^2+\dot{g}^2)\right\}. \tag{37}$$

If the developable surface given by equation (35) is biharmonic, i.e. if $\nabla^4 w = 0$, the coefficients of u^{-3}, u^{-4}, and u^{-5} must vanish independently. From the last term in equation (37) it is seen that the only possibility for this to be true is $\phi = 0$. But $\phi = 0$ implies $\psi = 0$, which in turn implies $\partial^2 w/\partial x^2 = \partial^2 w/\partial y^2 = 0$, and the surface must be a plane.

Developable surfaces that are not represented by equation (35) are cylinders and cones. It is easy to see that any cubic cylinder is both developable and biharmonic.

REFERENCES

1. H. Reissner, 'Spannungen in Kugelschalen-(Kuppeln)', *Festschrift Mueller-Breslau* (Leipzig, 1912), p. 181.
2. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed. (Cambridge, 1927), pp. 29, 459.
3. K. O. Friedrichs, 'The edge effect in the bending of plates', *Reissner Anniversary Volume*.
4. G. F. C. Searle, *Experimental Elasticity* (Cambridge, 1908), pp. 40–58.
5. J. Case, *Strength of Materials*, 3rd ed. (Arnold, 1938), pp. 546–54.
6. D. G. Ashwell, 'The anticlastic curvature of rectangular beams and plates', *J.R.Ae.S.* **54** (1950), 708.
7. —— and E. D. Greenwood, 'The pure bending of rectangular plates', *Engineering*, 21 and 28 July (1950).
8. —— 'A characteristic type of instability in the large deflexions of elastic plates. I. Curved rectangular plates bent about one axis. II. Flat square plates bent about all edges. *Proc. Roy. Soc.* A, **214** (1952), 98.
9. Kelvin and Tait, *A Treatise on Natural Philosophy*, 2nd ed. (Cambridge, 1890), § 656.
10. E. H. Mansfield and P. W. Kleeman, *A Large Deflexion Theory for Thin Plates*. R.A.E. Report Structures 153 (to be published), Oct. 1953.
11. —— —— *Stress Analysis of Triangular Cantilever Plates*. R.A.E. Report Structures 154 (to be published), Oct. 1953.
12. S. Timoshenko, *Theory of Elastic Stability* (McGraw-Hill, 1936).
13. E. Reissner, 'Pure bending and twisting of thin skewed plates', *Quarterly of Applied Mathematics*, **10** (1953), 395.
14. Th. von Kármán and H. S. Tsien, 'The buckling of thin cylindrical shells under axial compression', *J. Aero. Sci.* **8** (1941), 303.
15. D. M. A. Leggett and R. P. N. Jones, 'The behaviour of a cylindrical shell under axial compression when the buckling load has been exceeded', *A.R.C. R. & M.* 2190 (1942).
16. Y. Yoshimaru, 'On the mechanism of buckling of a circular cylindrical shell under axial compression', *Rep. Inst. Sci. and Tech., Tokyo Univ.* **5**, No. 5 (1951).
17. C. E. Weatherburn, *Differential Geometry of Three Dimensions* (Cambridge, 1930).
18. D. J. Struik, *Lectures on Classical Differential Geometry*. Addison-Wesley Press, Inc. (Cambridge, Mass., 1950).

THE TRANSFORMATION TO ISOTROPIC FORM OF THE EQUILIBRIUM EQUATIONS FOR A CLASS OF ANISOTROPIC ELASTIC SOLIDS

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SUMMARY

It is shown that the boundary conditions and differential equations governing the equilibrium of an anisotropic homogeneous elastic solid, whose 21 elastic constants satisfy 14 specially chosen conditions, can be expressed by means of a linear transformation in a form identical with the corresponding equations for an isotropic solid. This gives rise to a new method of treating equilibrium problems in the classical theory of elasticity for solids having anisotropy of a special kind. In the case of a solid which has three mutually orthogonal planes of elastic symmetry, 9 of the 14 conditions are automatically satisfied, and the transformation involves simple changes of scale in the symmetry directions; in the particular case of transverse isotropy, the number of conditions automatically satisfied is 12. The transformation can be applied to problems involving two or more elastic solids if they have the same elastic constants and are similarly oriented with regard to their directions of elastic symmetry. As an illustration, the transformation is used to derive, from Hertz's solution for the isotropic case, expressions giving the shape and size of the contact region formed when an elastic solid having a special kind of elastic anisotropy is pressed lightly against a rigid plane.

In the case of a transversely isotropic solid, it is shown (without restriction on the elastic constants) that the class of solutions of the differential equations of equilibrium obtainable by a method due to Elliott may be enlarged by using three stress functions of harmonic type instead of two. This leads to an interesting alternative derivation of the above restrictions on the elastic constants for the case of transverse isotropy.

1. Introduction

In considering the solution of a problem involving the contact of transversely isotropic elastic solids it was found that a considerable simplification resulted when the five elastic constants involved were taken to satisfy two conditions. When these conditions were introduced in the general equilibrium equations of elasticity theory, it was found that these simplified also, and were expressible in terms of new variables in a form identical with that of the corresponding equations for isotropic solids. Any given problem involving homogeneous solids with the right kind of anisotropy could thereby be transformed into an isotropic problem, in which form it should be less difficult to solve. A brief note of this result has been published elsewhere (**1**). The object of the present paper is to extend these transformations to include as wide as possible a class of anisotropic solids.

It is found that even with the most general linear transformation (the variables being referred to a rectangular Cartesian coordinate system), the 21 elastic constants for the general solid must satisfy 14 conditions, so that only 7 constants are independent. For solids having orthorhombic elastic symmetry all but 5 of the conditions are automatically satisfied, and there are 4 independent constants; for transversely isotropic solids all but 2 of the conditions are automatically satisfied, and there are 3 independent constants, as mentioned above. Since non-linear transformations are unlikely to be of interest, the extension given here probably represents the best that can be achieved by the present method.

The following assumptions and restrictions are made throughout:

(i) The solids are ideally elastic, and the strains are infinitesimally small (as in the classical theory of elasticity).

(ii) The elastic constants satisfy equations (11) below [or (16) for orthorhombic symmetry, (26) for transverse isotropy].

(iii) Only equilibrium problems can be dealt with; motion and vibration are excluded.

(iv) The elastic solids are homogeneous. When two or more such solids are involved in one problem, they must have the same elastic constants and the same directions of elastic symmetry. The homogeneity restriction can be relaxed with regard to two of the constants, but this is not likely to be of interest in practice.

The method is based on the use of new dependent and independent variables. The ones used in the present paper involve the choice of a special rectangular Cartesian coordinate system (for the case of an orthorhombic solid, the coordinate planes must be parallel to the planes of elastic symmetry), to which the equations of any given problem must be referred before the transformations given in the paper can be applied. The use of a special coordinate system is not, however, essential to the method; the actual transformations to be applied in any other coordinate system could, if required, be derived from those given in this paper.

In section 2 the basic transformations of the differential equations of equilibrium are derived, together with the restrictions which have to be imposed on the elastic constants. In section 3 the special case of solids with orthorhombic elastic symmetry is treated, a summary of the results being given in section 4 in a form convenient for application to actual problems. In section 5 the special case of transversely isotropic solids is treated, and a digression is made in order to give a generalization of Elliott's transformation. In section 6 the transformation of boundary conditions is considered briefly, and in section 7 the methods are applied to obtain a

generalization of Hertz's solution to a problem involving the contact of elastic solids. The paper ends (section 8) with a discussion of the applicability of the methods to actual materials.

2. Transformation of the differential equations

In the classical theory of elasticity the partial differential equations which have to be solved to determine the equilibrium state of an anisotropic elastic solid subjected to given forces and constraints may be written in the form

$$\frac{\partial p_{ir}}{\partial x_r}+\rho X_i = 0, \tag{1}$$

$$2p_{ij} = \frac{\partial W}{\partial e_{ij}}+\frac{\partial W}{\partial e_{ji}}, \tag{2}$$

$$2e_{ij} = \frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}, \tag{3}$$

$$\begin{aligned} W = {} & \tfrac{1}{2}c_{11}e_{11}^2+c_{12}e_{11}e_{22}+c_{13}e_{11}e_{33}+2e_{11}(c_{14}e_{23}+c_{15}e_{31}+c_{16}e_{21})+ \\ & +\tfrac{1}{2}c_{22}e_{22}^2+c_{23}e_{22}e_{33}+2e_{22}(c_{24}e_{23}+c_{25}e_{31}+c_{26}e_{21})+ \\ & +\tfrac{1}{2}c_{33}e_{33}^2+2e_{33}(c_{34}e_{23}+c_{35}e_{31}+c_{36}e_{21})+ \\ & +2e_{23}(c_{44}e_{23}+2c_{45}e_{31}+2c_{46}e_{21})+ \\ & +2c_{55}e_{31}^2+4c_{56}e_{31}e_{12}+2c_{66}e_{12}^2, \end{aligned} \tag{4}$$

where p_{ij}, e_{ij}, x_i, u_i, X_i, ρ, W denote the stress components, infinitesimal strain components, coordinates, displacement components, components of body force per unit mass, density, and strain-energy per unit mass. The coordinate system is rectangular Cartesian fixed in space. Latin suffixes take on the values 1, 2, and 3, and each term containing a repeated Latin suffix is to be understood, here and throughout, as summed for the three values of that suffix, unless the contrary is stated.

The coefficients $c_{\alpha\beta}$ (α, $\beta = 1, 2,\ldots, 6$), which are constant for a homogeneous material, characterize the elastic properties of the material; for the most general type of anisotropy, 21 of them are independent (cf. (**2**), p. 159, allowing for differences in notation, namely $e_{xx} = e_{11}$, $e_{yz} = 2e_{23}$, etc.; our $c_{\alpha\beta}$ are the same as Love's).

The differential equations are linear; in certain circumstances there exists a linear change of variables of the form

$$x_i \to x'_i, \quad u_i \to u'_i, \quad e_{ij} \to e'_{ij}, \quad p_{ij} \to p'_{ij}, \quad X_i \to X'_i, \tag{5}$$

where

$$\left.\begin{aligned} x_i &= a_{ri}x'_r \\ u'_i &= a_{ir}u_r \\ e'_{ij} &= a_{ir}a_{js}e_{rs} \\ p_{ij} &= a_{ri}a_{sj}p'_{rs} \\ X_i &= a_{ri}X'_r \end{aligned}\right\}, \tag{6}$$

the a_{ij} being constants satisfying the condition

$$\det(a_{ij}) = 0 \tag{7}$$

for linear independence of the x_i, which transforms the differential equations into a form identical with that of the corresponding equations for an isotropic solid: equations of the same form as equations (1), (2), and (3) are obtained, while the strain-energy function (4) takes the form

$$W = (\tfrac{1}{2}\lambda\delta_{ij}\,\delta_{rs}+\mu\delta_{ir}\,\delta_{js})e'_{ij}\,e'_{rs} \tag{8}$$

appropriate for an isotropic solid of Lamé elastic constants λ, μ. δ_{ij} equals 1 or 0 according as i, j are equal or unequal.

In order to verify this statement, perhaps the simplest procedure is to start with the isotropic equations and work backwards. We need only outline the proof, which is straightforward, though lengthy. The three equations (1), (2), and (3) are found to be invariant in form under the transformation (5), whatever the values of the transformation coefficients a_{ij} may be. Application of the inverse of transformation (5) to the isotropic form of the strain-energy function (8) leads to an expression which may be written in the form

$$W = (\tfrac{1}{2}\lambda A_{ij}\,A_{rs}+\mu A_{ir}\,A_{js})e_{ij}\,e_{rs}, \tag{9}$$

where

$$A_{ij} = a_{ri}\,a_{rj}. \tag{10}$$

This expression, which involves only 7 independent constants (6 independent A_{ij} and the ratio λ/μ, for example), has to be identified with the strain-energy function (4) for the general anisotropic material. This is possible when (and only when) the 21 constants $c_{\alpha\beta}$ in the latter satisfy certain conditions, 14 in number, which are found to be expressible as follows:

$$\begin{aligned}
2c_{44}+c_{23} &= (c_{22}\,c_{33})^{\frac{1}{2}}+4c_{24}^2/c_{22},\\
2c_{55}+c_{31} &= (c_{33}\,c_{11})^{\frac{1}{2}}+4c_{35}^2/c_{33},\\
2c_{66}+c_{12} &= (c_{11}\,c_{22})^{\frac{1}{2}}+4c_{16}^2/c_{11},\\
2c_{56}+c_{14} &= c_{24}(c_{11}/c_{22})^{\frac{1}{2}}+2c_{35}\,c_{16}(c_{33}\,c_{11})^{-\frac{1}{2}},\\
2c_{64}+c_{25} &= c_{35}(c_{22}/c_{33})^{\frac{1}{2}}+2c_{16}\,c_{24}(c_{11}\,c_{22})^{-\frac{1}{2}},\\
2c_{45}+c_{36} &= c_{16}(c_{33}/c_{11})^{\frac{1}{2}}+2c_{24}\,c_{35}(c_{22}\,c_{33})^{-\frac{1}{2}},\\
c_{15}/c_{35} &= (c_{11}/c_{33})^{\frac{1}{2}},\\
c_{26}/c_{16} &= (c_{22}/c_{11})^{\frac{1}{2}},\\
c_{34}/c_{24} &= (c_{33}/c_{22})^{\frac{1}{2}},
\end{aligned}$$

$$\frac{(c_{22}c_{33})^{\frac{1}{2}}-4c_{24}^2/c_{22}}{c_{23}-4c_{24}^2/c_{22}}=\frac{(c_{33}c_{11})^{\frac{1}{2}}-4c_{35}^2/c_{33}}{c_{31}-4c_{35}^2/c_{33}}=\frac{(c_{11}c_{22})^{\frac{1}{2}}-4c_{16}^2/c_{11}}{c_{12}-4c_{16}^2/c_{11}}$$
$$=\frac{c_{24}(c_{11}/c_{22})^{\frac{1}{2}}-2c_{35}c_{16}(c_{33}c_{11})^{-\frac{1}{2}}}{c_{14}-2c_{35}c_{16}(c_{33}c_{11})^{-\frac{1}{2}}}=\frac{c_{35}(c_{22}/c_{33})^{\frac{1}{2}}-2c_{16}c_{24}(c_{11}c_{22})^{-\frac{1}{2}}}{c_{25}-2c_{16}c_{24}(c_{11}c_{22})^{-\frac{1}{2}}}$$
$$=\frac{c_{16}(c_{33}/c_{11})^{\frac{1}{2}}-2c_{24}c_{35}(c_{22}c_{33})^{-\frac{1}{2}}}{c_{36}-2c_{24}c_{35}(c_{22}c_{33})^{-\frac{1}{2}}} \tag{11}$$
$$=1+2\mu/\lambda. \tag{12}$$

Equations (11) are therefore a necessary and sufficient set of conditions for the transformation to an isotropic form by means of a transformation of the form (5), (6) to be possible. This form can be shown to be the most general linear one which leaves the differential equations (1), (2), and (3) invariant. The positive signs have to be taken for the radicals in (11) (and throughout the paper) in order to give a real transformation (6).

In order to make use of the transformation in an actual problem, we have to be able to express the transformation coefficients a_{ij} in terms of the given elastic constants $c_{\alpha\beta}$. The relations required for this may be obtained by identifying the two forms (4), (9) for the strain-energy function, and are found to be expressible as follows:

$$\left.\begin{aligned}A_{11}&=[c_{11}/(\lambda+2\mu)]^{\frac{1}{2}}\\A_{22}&=[c_{22}/(\lambda+2\mu)]^{\frac{1}{2}}\\A_{33}&=[c_{33}/(\lambda+2\mu)]^{\frac{1}{2}}\\A_{23}&=2c_{24}[c_{22}(\lambda+2\mu)]^{-\frac{1}{2}}\\A_{31}&=2c_{35}[c_{33}(\lambda+2\mu)]^{-\frac{1}{2}}\\A_{12}&=2c_{16}[c_{11}(\lambda+2\mu)]^{-\frac{1}{2}}\end{aligned}\right\}. \tag{13}$$

The elastic constants λ, μ, of what may be called the 'equivalent isotropic material', are determined only in so far as their ratio is given by equation (12), and they may be given any values consistent with this equation. There is also a certain freedom of choice in the values to be given to the 9 transformation coefficients a_{ij}, because they are determined only by equations (13), where they occur in only 6 independent combinations A_{ij}.

The transformation as given above has been restricted to materials which are homogeneous (i.e. the $c_{\alpha\beta}$ are constants). This restriction is likely to be most felt in those problems which involve more than one elastic body, for it means that the bodies must not only have the same elastic constants but must also be similarly oriented with regard to their directions of elastic symmetry. A slight, but probably unimportant, relaxation of this homogeneity requirement can be made, because the $c_{\alpha\beta}$ can be regarded as functions of a_{ij}, λ, and μ, and it is the transformation coefficients a_{ij} which

have to be constant, not λ and μ, which can be functions of x_i, i.e. the equivalent isotropic material need not be homogeneous.

It has been stated that the transformation to isotropic form is possible only for equilibrium problems, but in view of the importance of elastic wave propagation in anisotropic materials, it is worth noting what happens under the above transformation when the inertial term for small motions, $\rho\,\partial^2 u_i/\partial t^2$, is added to the right-hand side of equation (1). The transformed equation is found to be

$$\frac{\partial p_{ir}}{\partial x_r'}+\rho X_i' = \rho\frac{\partial^2 u_r'}{\partial t^2}A_{ri}. \tag{14}$$

When the inertial term is not zero, it is seen that the form of this equation is invariant only when $A_{ij}=\delta_{ij}$, which corresponds to the degenerate case in which the original material is isotropic and the transformation can be interpreted simply as a change from one rectangular Cartesian coordinate system to another.

Some (but not all) of the 14 conditions (11) on the elastic constants $c_{\alpha\beta}$ are automatically satisfied when the material possesses lines or planes of elastic symmetry. Two such cases will now be considered.

3. Solids with orthorhombic elastic symmetry

For a material with orthorhombic symmetry or with three mutually orthogonal planes of symmetry, 14 of the 21 elastic constants $c_{\alpha\beta}$ are zero, and the strain-energy function may be written in the form

$$W = \tfrac{1}{2}(c_{11}e_{11}^2+c_{22}e_{22}^2+c_{33}e_{33}^2)+c_{23}e_{22}e_{33}+c_{31}e_{33}e_{11}+c_{12}e_{11}e_{22}+ \\ +2(c_{44}e_{23}^2+c_{55}e_{31}^2+c_{66}e_{12}^2), \tag{15}$$

the coordinate planes being taken parallel to the symmetry planes (**2**, p. 159).

In this case it is found that 9 of our 14 special conditions (11) are satisfied, leaving 5, which may be expressed in the form

$$\left.\begin{aligned} 2c_{44}+c_{23} &= (c_{22}c_{33})^{\frac{1}{2}} \\ 2c_{55}+c_{31} &= (c_{33}c_{11})^{\frac{1}{2}} \\ 2c_{66}+c_{12} &= (c_{11}c_{22})^{\frac{1}{2}} \\ (c_{22}c_{33})^{\frac{1}{2}}/c_{23} = (c_{33}c_{11})^{\frac{1}{2}}/c_{31} &= (c_{11}c_{22})^{\frac{1}{2}}/c_{12} \quad [=1+2\mu/\lambda] \end{aligned}\right\} \tag{16}$$

in terms of the $c_{\alpha\beta}$, or, alternatively, in the form

$$\left.\begin{aligned} s_{23}+\tfrac{1}{2}s_{44} &= (s_{22}s_{33})^{\frac{1}{2}} \\ s_{31}+\tfrac{1}{2}s_{55} &= (s_{33}s_{11})^{\frac{1}{2}} \\ s_{12}+\tfrac{1}{2}s_{66} &= (s_{11}s_{22})^{\frac{1}{2}} \\ (s_{22}s_{33})^{\frac{1}{2}}/s_{23} = (s_{33}s_{11})^{\frac{1}{2}}/s_{31} &= (s_{11}s_{22})^{\frac{1}{2}}/s_{12} \quad [=-\sigma^{-1}] \end{aligned}\right\}, \tag{17}$$

in terms of the elastic moduli $s_{\alpha\beta}$, which are related to the $c_{\alpha\beta}$ by equations of the type

$$s_{11}\Delta = c_{22}c_{33}-c_{23}^2; \qquad s_{23}\Delta = c_{21}c_{13}-c_{11}c_{23}; \qquad s_{44} = c_{44}^{-1}; \text{ etc.}, \tag{18}$$

where

$$\Delta = \det(c_{ij}) \quad (i, j = 1, 2, 3),$$

and to the Young's moduli E_i, shear moduli G_i, and Poisson's ratios σ_{ij}, by equations of the type

$$E_1 = s_{11}^{-1}, \qquad G_1 = s_{44}^{-1}, \qquad \sigma_{12} = -s_{12}/s_{11}, \qquad \text{etc.} \tag{19}$$

E_1 and σ_{12} denote Young's modulus and a Poisson's ratio for a tensile stress in the direction Ox_1, and G_1 denotes the rigidity modulus for a shear in which planes normal to Ox_2 move in the direction Ox_3; the other symbols have similar definitions and $\sigma = \frac{1}{2}\lambda/(\lambda+\mu)$ is the Poisson's ratio of the equivalent isotropic material.

Equations (16) or (17) are thus the conditions to be satisfied in the case of an orthorhombic solid if the transformation to isotropic form is to be possible. These conditions have some direct physical significance, for the first three of equations (17) are the conditions for the polar diagram of the reciprocal of the fourth root of Young's modulus to be ellipsoidal (instead of a more complicated quartic surface (**2**, p. 161)), while the two remaining equations can be expressed in terms of Poisson's ratios in the form

$$\sigma_{23}\sigma_{32} = \sigma_{31}\sigma_{13} = \sigma_{12}\sigma_{21} \quad [= \sigma^2]. \tag{20}$$

Another form of the conditions which is perhaps worth noting is the following:

$$\left.\begin{aligned} \sigma_{13}/\sigma_{12} &= G_2/G_3 = (E_2/E_3)^{\frac{1}{2}} \\ \sigma_{21}/\sigma_{23} &= G_3/G_1 = (E_3/E_1)^{\frac{1}{2}} \\ \sigma_{32}/\sigma_{31} &= G_1/G_2 = (E_1/E_2)^{\frac{1}{2}} \end{aligned}\right\}. \tag{21}$$

These are necessary but not sufficient, only four of them being independent.

In the case of orthorhombic symmetry the transformation coefficients a_{ij} can be given the simple set of values

$$\left.\begin{aligned} a_{ij} &= 0 \quad (i \neq j), \\ a_{ii} &= (E_i/E)^{\frac{1}{4}} \quad (i \text{ not summed}), \end{aligned}\right\} \tag{22}$$

where E is an arbitrary positive constant, equal to the Young's modulus of the equivalent isotropic material. This follows from equations (10), (13), (16), (18), (19), and the known relation $E = \mu(3\lambda+2\mu)/(\lambda+\mu)$.

In the next section we collect together all the results, for the case of orthorhombic symmetry, which will be needed in applying the present transformation methods to any actual problem.

4. Summary of transformations for the case of orthorhombic symmetry

In the differential equations of equilibrium, referred to a rectangular Cartesian coordinate system whose coordinate planes are parallel to the planes of elastic symmetry, let the variables

$$x_i, \quad u_i, \quad e_{ij}, \quad p_{ij}, \quad X_i \quad (i, j = 1, 2, 3), \tag{23}$$

representing respectively the coordinates, displacement components, infinitesimal strain components, stress components, and components of body force per unit mass, be replaced by the quantities

$$\left(\frac{E_i}{E}\right)^{\frac{1}{4}} x_i', \quad \left(\frac{E}{E_i}\right)^{\frac{1}{4}} u_i', \quad \left(\frac{E^2}{E_i E_j}\right)^{\frac{1}{4}} e_{ij}', \quad \left(\frac{E_i E_j}{E^2}\right)^{\frac{1}{4}} p_{ij}', \quad \left(\frac{E_i}{E}\right)^{\frac{1}{4}} X_i' \ (i, j \text{ not summed}), \tag{24}$$

where E_i are the Young's moduli in the symmetry directions, and E is an arbitrary positive constant. If the elastic constants of the solid satisfy equations (16) (or (17)), the resulting equations in the new variables x_i', u_i', etc., will have the same form as the corresponding equations for an isotropic elastic solid expressed in terms of the usual variables, x_i, u_i, etc.

The Young's modulus of this 'equivalent isotropic solid' is equal to the arbitrary constant E; the Poisson's ratio σ is determined by the Poisson's ratios σ_{ij} of the given anisotropic solid according to the equations

$$\sigma^2 = \sigma_{23}\sigma_{32} = \sigma_{31}\sigma_{13} = \sigma_{12}\sigma_{21}. \tag{20}$$

5. Transversely isotropic solids

A particular case of the class of orthorhombic solids worth special mention is the class of transversely isotropic solids, that is, solids which have an axis of elastic symmetry. The strain-energy function contains five independent constants and may be obtained from the strain-energy function (15) for an orthorhombic solid by putting

$$c_{11} = c_{22}, \qquad c_{23} = c_{31}, \qquad c_{55} = c_{44}, \qquad c_{66} = \tfrac{1}{2}(c_{11}-c_{12}), \tag{25}$$

when the x_3-axis is taken parallel to the axis of elastic symmetry (cf. (2), p. 160, groups C_3^h, etc.).

In order that the transformation to isotropic form may be carried out for a transversely isotropic solid, its elastic constants must satisfy two conditions, viz.

$$c_{13}+2c_{44} = (c_{11}c_{33})^{\frac{1}{2}} = c_{31}c_{11}/c_{12}, \tag{26}$$

which may be obtained from equations (16) and (25). Other sets of necessary and sufficient conditions may be derived from those given in section 3 for orthorhombic solids; we mention here only one set, namely that the polar

diagram of the reciprocal of the fourth root of Young's modulus is an ellipsoid of revolution about the axis of elastic symmetry, and

$$\sigma_{31}\sigma_{13} = \sigma_{12}^2. \tag{27}$$

This follows from equation (20) and the fact that, owing to the transverse isotropy, $\sigma_{12} = \sigma_{21}$, $\sigma_{32} = \sigma_{31}$, and $\sigma_{23} = \sigma_{13}$.

The changes of variables which enable one to effect the transformation to isotropic form in the case of transverse isotropy may be readily obtained by putting $E_1 = E_2$ in those given in section 4 for the case of orthorhombic symmetry, and need not be given separately.

We now make a brief digression to state a new result in the general theory of transversely isotropic elastic solids which has some connexion with the main topic of the present paper. For a transversely isotropic solid of general type (whose elastic constants do not necessarily satisfy our special conditions (26)), it is a straightforward matter to verify that the differential equations of equilibrium are satisfied identically when the displacement components u_i are expressed in the form

$$\left.\begin{aligned} u_1 &= \frac{\partial}{\partial x_1}(\phi_1+\phi_2)+\frac{\partial \phi_3}{\partial x_2} \\ u_2 &= \frac{\partial}{\partial x_2}(\phi_1+\phi_2)-\frac{\partial \phi_3}{\partial x_1} \\ u_3 &= \frac{\partial}{\partial x_3}(k_1\phi_1+k_2\phi_2) \end{aligned}\right\}, \tag{28}$$

in terms of three functions ϕ_i which satisfy the equations

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\nu_i\frac{\partial^2}{\partial x_3^2}\right)\phi_i = 0 \quad (i = 1, 2, 3, \text{ not summed}) \tag{29}$$

but are otherwise arbitrary.

Here, ν_1 and ν_2 denote the roots in ν of the equation

$$c_{11}c_{44}\nu^2+(c_{13}^2-c_{11}c_{33}+2c_{13}c_{44})\nu+c_{33}c_{44} = 0, \tag{30}$$

$$\nu_3 = 2c_{44}/(c_{11}-c_{22}), \tag{31}$$

and

$$k_\alpha = (c_{11}\nu_\alpha-c_{44})/(c_{13}+c_{44}) \quad (\alpha = 1, 2). \tag{32}$$

This is an extension of a result due to Elliott (**3**), who gave the particular case $\phi_3 = 0$ appropriate to problems in which the component of rotation $(\partial u_1/\partial x_2-\partial u_2/\partial x_1)$ about the direction of elastic symmetry is everywhere zero. The present result will be applicable to a wider class of problems, but whether it is completely general or not is not known.

It is instructive to relate this result to the transformation method of the present paper which applies to transversely isotropic solids whose elastic

constants satisfy equations (26). These equations are equivalent to the equations

$$\nu_1 = \nu_2 = \nu_3, \tag{33}$$

as can readily be verified using (30), (31). When these equations are satisfied, it is obvious that the differential equations (29) can be put into isotropic form by the transformation

$$x_1 = x'_1, \qquad x_2 = x'_2, \qquad x_3 = \nu_3^{\frac{1}{2}} x'_3. \tag{34}$$

We now return to the main topic of the paper.

6. Transformation of boundary conditions

We have seen that for suitable types of anisotropic elastic material the differential equations of equilibrium can be put into isotropic form, for which many solutions are known. In any specific problem it is necessary to find a solution which fits boundary conditions characteristic of the problem which may be expressed in or derived from one of two general forms, namely

$$u_i = u_i^{(0)} \quad \text{on} \quad s(x) = \text{const.}\dagger \tag{35}$$

when the displacement components have prescribed values on the surface of the body

$$s(x) = \text{const.}, \tag{36}$$

or

$$p_{ij} n_j = Y_i \quad \text{on} \quad s(x) = \text{const.} \tag{37}$$

when the components of surface force per unit area have prescribed values Y_i. Let n_i be the components of the unit normal to the surface; these are given by the equations

$$n_i = \frac{\partial s}{\partial x_i}\left[\frac{\partial s}{\partial x_r}\frac{\partial s}{\partial x_r}\right]^{-\frac{1}{2}}. \tag{38}$$

It is easy to verify that, if we apply to these equations the transformation (6) which we applied to the differential equations, equations of the same form in the primed variables are obtained, provided we define the primed function s' by the identity

$$s'(x') = s(x), \tag{39}$$

and the primed variables Y'_i by the equations

$$Y_i = \eta a_{ri} Y'_r, \tag{40}$$

where

$$\eta = (A_{rs} n_r n_s)^{\frac{1}{2}}. \tag{41}$$

It follows that the equivalent isotropic problem is a problem of the same kind as the given anisotropic problem, the possible differences being in the shape of the bounding surface, the distribution of surface forces or displacements, and the distribution of body forces. General methods of solution available in isotropic elasticity theory may therefore be used; the isotropic solution may even be known already, in which case the only labour

† x is an abbreviation for x_1, x_2, x_3.

involved in solving the given anisotropic problem arises from making the appropriate transformation in the isotropic solution. An example of this will now be given.

7. Example: contact of an elastic solid with a rigid plane

As an illustration of the use of the above method in the solution of actual problems, we shall now use it to determine the shape and size of the contact region formed when an elastic solid with a curved surface is pressed lightly against a rigid plane with a given normal force P (say). Hertz's solution for the isotropic case is well known, and we use it to derive the solution for a homogeneous anisotropic solid having orthorhombic symmetry of the special type considered above, i.e. the elastic constants are taken to satisfy equations (16) or (17) in addition to the conditions of orthorhombic symmetry. For simplicity, we assume that the planes of elastic symmetry are parallel to the planes of principal curvature and the tangent plane at the point at which contact is to be made.

As in Hertz's theory, we assume that the surface of the elastic solid is convex to the plane and sufficiently regular near the point at which contact is to be made to be represented there by an equation of the form

$$x_2^2/r_2+x_3^2/r_3 = 2x_1, \tag{42}$$

where r_2, r_3 are the principal radii of curvature; the coordinate system is chosen so that the x_1-axis is normal to the rigid plane at the point of contact and the other two axes lie in the rigid plane and are parallel to the principal directions of curvature.

When the force is applied we shall assume that the contact region formed is bounded by an ellipse

$$x_2^2/b^2+x_3^2/c^2 = 1, \qquad x_1 = 0, \tag{43}$$

whose semi-axes b, c are to be determined. As in the isotropic theory, this assumption will prove to be justified because it enables us to find a solution satisfying all the conditions of the problem.

As in Hertz's theory, we assume that the contact region is so small compared with the rest of the elastic body that this may be treated as if it were infinite in extent in formulating the boundary conditions. These conditions may then be written as follows:

$$\left.\begin{array}{l}
-u_1 \overset{=}{>} x_2^2/r_2+x_3^2/r_3 \quad \text{when } x_2^2/b^2+x_3^2/c^2 \overset{\leqslant}{>} 1,\ x_1 = 0. \\
u_2 = u_3 = 0, \quad u_1 = d \text{ (a constant)}, \quad \text{when } x = \infty. \\
\text{All } p_{ij} = 0 \quad \text{when } x_2^2/b^2+x_3^2/c^2 > 1,\ x_1 = 0. \\
\text{All } p_{ij} \text{ (except } p_{11}) = 0 \quad \text{when } x_2^2/b^2+x_3^2/c^2 \leqslant 1,\ x_1 = 0. \\
\iint p_{11}\,dx_2\,dx_3 = P, \quad \text{the integration being taken over the} \\
\qquad \text{contact region } x_3^2/b^2+x_3^2/c^2 \leqslant 1,\ x_1 = 0.
\end{array}\right\} \tag{44}$$

These are the boundary conditions for the given anisotropic problem. To obtain from them the boundary conditions for the equivalent isotropic problem we replace the variables (23), wherever they occur in (44), by the variables (24). We then find that equations are obtained having the same form as the given boundary conditions (44) but with the constants P, r_2, r_3, b, c, d replaced by P', r_2', r_3', b', c', d', where

$$\left.\begin{aligned} P' &= P\left(\frac{E}{E_1}\right)^{\frac{1}{2}}\left(\frac{E^2}{E_2 E_3}\right)^{\frac{1}{4}} \\ r_2' &= r_2\left(\frac{E}{E_1}\right)^{\frac{1}{4}}\left(\frac{E}{E_2}\right)^{\frac{1}{2}} \\ r_3' &= r_3\left(\frac{E}{E_1}\right)^{\frac{1}{4}}\left(\frac{E}{E_3}\right)^{\frac{1}{2}} \\ b' &= b\left(\frac{E}{E_2}\right)^{\frac{1}{4}} \\ c' &= c\left(\frac{E}{E_3}\right)^{\frac{1}{4}} \\ d' &= d\left(\frac{E_1}{E}\right)^{\frac{1}{4}} \end{aligned}\right\}. \tag{45}$$

From this it follows that the equivalent isotropic problem is a contact problem of the same kind as the given one, but with different values (P', r_2', r_3') for the applied force and the radii of curvature. The reason for this specially simple result is that the surface on which the boundary conditions are specified is a plane parallel to a plane of elastic symmetry.

The solution of the equivalent isotropic problem may therefore be taken directly from Hertz's solution. For convenience we use the form given by Timoshenko (**4**). Allowing for the fact that in our case one body is a rigid plane, and making the appropriate changes in notation (a, b, A, B, P, k_1, k_2 in Timoshenko's notation have to be replaced by c', b', $(2r_3')^{-1}$, $(2r_2')^{-1}$, P', $(1-\sigma^2)/\pi E$, 0 in our notation), it follows that the contact region is an ellipse whose semi-axes b', c' are given by the equations

$$\frac{b'}{n} = \frac{c'}{m} = \left\{\frac{3P'}{2}\left[\frac{1}{r_2'}+\frac{1}{r_3'}\right]^{-1}\frac{1-\sigma^2}{E}\right\}^{\frac{1}{3}}, \tag{46}$$

where m, n are functions, tabulated by Timoshenko, of a variable θ defined by the equation

$$\cos\theta = (r_3'-r_2')/(r_3'+r_2'). \tag{47}$$

To obtain the solution of the anisotropic problem we use (45) to express (46) and (47) in terms of P, r_2, r_3, b, c, and we use equation (20) to express σ

in terms of the Poisson's ratios of the anisotropic material. After a slight rearrangement we finally obtain the result that the contact area is bounded by an ellipse as assumed, and that the semi-axes b, c of this ellipse are given by the equations

$$\left(\frac{E_1}{E_2}\right)^{\frac{1}{4}}\frac{b}{n} = \left(\frac{E_1}{E_3}\right)^{\frac{1}{4}}\frac{c}{m} = \left\{\frac{3P}{2}\left[\frac{E_2^{\frac{1}{2}}}{r_2 E_1^{\frac{1}{2}}}+\frac{E_3^{\frac{1}{2}}}{r_3 E_1^{\frac{1}{2}}}\right]^{-1}\frac{1-\sigma_{23}\sigma_{32}}{(E_1^2 E_2 E_3)^{\frac{1}{4}}}\right\}^{\frac{1}{3}}, \tag{48}$$

where m, n are the functions of θ tabulated by Timoshenko, and

$$\cos\theta = (r_3 E_3^{-\frac{1}{2}} - r_2 E_2^{-\frac{1}{2}})/(r_3 E_3^{-\frac{1}{2}} + r_2 E_2^{-\frac{1}{2}}). \tag{49}$$

This completes the required solution, which I believe to be new.

As a check on these results, we may consider the particular case in which the elastic solid is transversely isotropic in planes parallel to the plane of contact ($x_1 = 0$). In this case we have $E_2 = E_3$ and $\sigma_{23} = \sigma_{32}$, so that our results (48), (49) give

$$\frac{b}{n} = \frac{c}{m} = \left[\frac{E_2}{E_1}\right]^{\frac{1}{4}}\left\{\frac{3P}{2}\left[\frac{1}{r_2}+\frac{1}{r_3}\right]^{-1}\frac{1-\sigma_{23}^2}{E_2}\right\}^{\frac{1}{3}}, \tag{50}$$

$$\cos\theta = (r_3 - r_2)/(r_3 + r_2). \tag{51}$$

On the other hand, the solution of the same problem for a transversely isotropic solid of general type may be obtained independently by a straightforward calculation of Love's type (**2**, p. 193), based on Michell's solution (**5**) for the corresponding 'problem of the plane'; when our special restrictions (similar to (26), but with the x_1-axis as axis of symmetry) on the elastic constants are introduced, the results obtained agree with (50).

For a transversely isotropic solid in which the direction of elastic symmetry is not perpendicular to the contact plane (in contrast to the last example) but parallel to it, the contact problem loses its symmetry and known methods do not apply; the present methods were in fact developed in an attempt to deal with this case. Taking the x_3-axis to be parallel to the direction of elastic symmetry, we have $E_1 = E_2$ and $\sigma_{23}\sigma_{32} = \sigma_{12}^2$ (from (27)), and the solution is found from that for the orthorhombic case ((48), (49)) to be given by the equations

$$\frac{b}{n} = \frac{c}{m\nu^{\frac{1}{2}}} = \left\{\frac{3P}{2}\left[\frac{1}{r_2}+\frac{\nu}{r_3}\right]^{-1}\frac{1-\sigma_{12}^2}{(E_1^3 E_3)^{\frac{1}{4}}}\right\}^{\frac{1}{3}}, \tag{52}$$

$$\cos\theta = (\nu^{-1} r_3 - r_2)/(\nu^{-1} r_3 + r_2), \tag{53}$$

where $\nu = (E_3/E_1)^{\frac{1}{2}}$ may be taken as a measure of the anisotropy of the material.

8. Discussion

The limitations of the present method arise from the special restrictions on the elastic constants which are imposed by the mathematical simplification of the differential equations of the theory and may not be satisfied by any actual materials. If there are in fact any materials which satisfy the conditions, it is likely that they will be orthorhombic or transversely isotropic, because such materials automatically satisfy most of the required conditions. However, none of the varieties of wood or the metallic crystals in these classes whose elastic constants are given in Hearmon's comprehensive list (**6**) satisfy all the required conditions.

The conditions consist of relations between physical constants, and it is natural to ask whether they have any special physical significance that indicates the kind of materials which might be expected to satisfy the conditions. For orthorhombic materials, three out of the five necessary conditions are equivalent to the condition that the polar diagram of the inverse fourth root of Young's modulus shall be an ellipsoid (whose principal axes are normal to the planes of elastic symmetry); in an isotropic material this polar diagram is a sphere. One might expect such a condition to be satisfied in materials like synthetic textile fibres whose elastic anisotropy is due to their having been permanently deformed by a pure, homogeneous deformation from an initial, elastically isotropic, state in which their constituent molecules are randomly oriented and distributed. There are, however, not enough experimental data available to test this rather tentative suggestion. Even if this condition is satisfied, there remain two other conditions (between the Poisson's ratios), and we have not found any reason for expecting them to be satisfied. It is a curious fact that one of the zinc samples in Hearmon's list satisfies the ellipsoidal polar diagram condition very accurately.

For materials which do not satisfy the required conditions the present methods might give a rough picture of the stress distribution and displacement field in problems which have not been solved exactly by other means. Such a picture might be of use for highly anisotropic materials which may come nearer to satisfying our conditions than the conditions of isotropy. In particular problems a knowledge of this rough solution might give a useful lead to the form of the exact solution.

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REFERENCES

1. A. S. Lodge, *Nature*, **169** (1952), 926.
2. A. E. H. Love, *The Mathematical Theory of Elasticity*, 4th ed. (Cambridge, 1927).
3. H. A. Elliott, *Proc. Camb. Phil. Soc.* **44** (1948), 522.
4. S. Timoshenko, *Theory of Elasticity*, 1st ed. (McGraw-Hill, 1934), p. 347.
5. J. H. Michell, *Proc. Lond. Math. Soc.* **32** (1900), 247.
6. R. F. S. Hearmon, *Rev. Mod. Phys.* **18** (1946), 409.

GENERALIZED PLANE STRESS IN AN ELASTIC WEDGE UNDER ISOLATED LOADS

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SUMMARY

General direct methods of solution are provided for both the stress and displacement in a wedge under force or couple nuclei acting at any point. The method is to adapt Tranter's Mellin transform treatment of the stresses to the complex variable approach and to extend it to deal with the displacement. Solutions are then given for a wedge under an internal couple nucleus with stress-free or rigid boundaries.

1. Introduction

A TREATMENT of generalized plane stress expressing the mean stresses and displacements in terms of complex potentials has been given by Stevenson (**1**) and here we shall follow the notation of that paper. In particular, complex potentials giving rise to force and couple nuclei are also given by Stevenson (**2**), but it appears impossible to find complex potentials in algebraic form which would remove the resulting flank stresses over the wedge boundaries. Tranter (**3**) has developed a Mellin transform treatment to find the Airy stress function for wedge problems with a distributed flank loading. This, with some modifications to suit the complex potential method, has been extended to include a treatment for the displacements. It will be shown here that the usual displacement function can be side-tracked and the displacements derived directly via the Mellin inversion integral, so that they involve only the Airy stress function transform. Solutions for normal and tangential point loading on the flanks were given by Shepherd (**4**) by Airy stress-function methods, but no attention appears to have been given hitherto to the problem of force or couple nuclei in the material of the wedge. The object of this paper is therefore to put forward a definite method of solution for these problems.

2. Notation

We shall use the mean stress combinations in polar coordinates r, θ given by $z = x+iy = re^{i\theta}$,

$$\Theta' = \breve{rr}+\breve{\theta\theta}, \qquad \Phi' = \breve{rr}-\breve{\theta\theta}+2i\breve{r\theta}, \tag{2.1}$$

which can be expressed in terms of complex potentials $\Omega(z)$, $\omega(z)$ in the forms

$$2\Theta' = \Omega'(z)+\overline{\Omega}'(\bar{z}), \tag{2.2}$$

$$-2\Phi' = \bar{z}\overline{\Omega}''(\bar{z})+\frac{\bar{z}}{z}\bar{\omega}''(\bar{z}) \tag{2.3}$$

in the absence of body force.

The complex mean displacement D' is given by

$$D' = U_r+iU_\theta = e^{-i\theta}D, \tag{2.4}$$

where the cartesian mean displacement D is given by

$$8\mu D = \kappa\Omega(z)-z\overline{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}). \tag{2.5}$$

Here μ is the rigidity and $\kappa = 3-4\sigma$, where σ is related to Poisson's ratio ν by

$$(1-\sigma)(1+\nu) = 1. \tag{2.6}$$

3. Force and couple nuclei

An isolated force singularity $F = F_x+iF_y$ acting at a point C, which is the origin of a complex variable z_1, is given by the complex potentials

$$\Omega_c(z_1) = -P\log z_1, \qquad \omega_c(z_1) = \kappa\overline{P}z_1\log z_1, \tag{3.1}$$

where

$$P = \frac{2F}{\pi(1+\kappa)}. \tag{3.2}$$

Similarly, an isolated couple singularity G may be represented by

$$\Omega_c(z_1) = 0, \qquad \omega_c(z_1) = \frac{2iG}{\pi}\log z_1. \tag{3.3}$$

For completeness in connexion with the problems detailed later, in which $z_1 = z-d$, it may be mentioned that these latter complex potentials give rise to stress and displacement combinations

$$\Theta' = 0, \qquad \Phi' = -\frac{iG}{\pi}\frac{1}{(r-de^{i\theta})^2}, \tag{3.4}$$

$$D' = \frac{iG}{4\pi\mu}\frac{1}{r-de^{i\theta}}. \tag{3.5}$$

4. The Mellin transform and its application to generalized plane stress

The Mellin transform $\overline{\overline{f(p)}}$ of the function $f(r)$ is defined by

$$\overline{\overline{f(p)}} = \int\limits_0^\infty r^{p-1}f(r)\,dr, \tag{4.1}$$

where p must be such that the integral converges. Provided that the integral $\int\limits_0^\infty r^{k-1}|f(r)|\,dr$ is bounded for some $k > 0$, an inversion integral

may be expressed in the form

$$f(r) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \overline{\overline{f(p)}}\, r^{-p}\, dp, \tag{4.2}$$

where $c > k > 0$.

It is proposed to apply this transform to the equations which express the stress components and displacements in terms of polar coordinates.

The mean stresses. The components $\breve{rr}$, $\breve{\theta\theta}$, $\breve{r\theta}$ are given in terms of Airy's stress function χ, assuming no body forces, by

$$\breve{rr} = \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\chi}{\partial r}; \qquad \breve{\theta\theta} = \frac{\partial^2\chi}{\partial r^2}; \qquad \breve{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right), \tag{4.3}$$

where

$$\nabla^4\chi = 0 \tag{4.4}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{4.5}$$

The mean displacements. Here U_r and U_θ are given (see Coker and Filon (**5**)) in terms of χ and a displacement function ψ satisfying

$$\nabla^2\psi = 0; \qquad \frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial\theta}\right) = \nabla^2\chi \tag{4.6}$$

as

$$\left.\begin{aligned} 2\mu U_r &= -\frac{\partial\chi}{\partial r}+(1-\sigma)r\frac{\partial\psi}{\partial\theta} \\ 2\mu U_\theta &= -\frac{1}{r}\frac{\partial\chi}{\partial\theta}+(1-\sigma)r^2\frac{\partial\psi}{\partial\theta} \end{aligned}\right\}. \tag{4.7}$$

Integration of (4.1) by parts gives the Mellin transform of derivatives in the form

$$\int\limits_0^\infty f^{(n)}(r)r^{p+n-1}\, dr = \frac{(-1)^n\Gamma(p+n)}{\Gamma(p)}\overline{\overline{f(p)}} \tag{4.8}$$

provided that

$$r^{p+m-1}f^{(m-1)}(r) \to 0 \quad \text{as } r\to 0 \text{ and as } r\to\infty \quad \text{for } m = 1, 2,\ldots, n. \tag{4.9}$$

Writing $D \equiv d/d\theta$ we may transform (4.4) and obtain

$$(D^2+p^2)\{D^2+(p+2)^2\}\overline{\overline{\chi}} = 0, \tag{4.10}$$

the solution of which is

$$\overline{\overline{\chi}} = Ae^{ip\theta}+\bar{A}e^{-ip\theta}+Be^{i(p+2)\theta}+\bar{B}e^{-i(p+2)\theta} \tag{4.11}$$

since it follows from the definition of $\overline{\overline{\chi}}$ that it is real if p is real.

The stress combinations may be similarly treated since

$$\Theta' = \nabla^2\chi; \qquad \Phi' = \left(\nabla^2-2\frac{\partial^2}{\partial r^2}\right)\chi-2i\frac{\partial}{\partial r}\left(\frac{1}{r}D\chi\right), \tag{4.12}$$

giving
$$\int_0^\infty \Theta' r^{p+1}\,dr = (D^2+p^2)\bar{\bar{\chi}} \tag{4.13}$$

and
$$\int_0^\infty \Phi' r^{p+1}\,dr = (D+ip)\{D+i(p+2)\}\bar{\bar{\chi}}. \tag{4.14}$$

From (4.9) when $m = 1$ it will be seen that we have here assumed that
$$r^{p+2}\Theta' \to 0 \quad \text{as } r \to 0 \text{ and as } r \to \infty.$$

Supposing $\Theta' = O(r^{-n})$ at infinity, we have, since it is finite at the origin,
$$-2 < p < n-2. \tag{4.15}$$

Transformation of the displacements may be accomplished by introducing an auxiliary displacement function ψ_1 given by
$$r^2\psi = \psi_1. \tag{4.16}$$

Equations (4.6) then take the form
$$\nabla^2\left(\frac{\psi_1}{r^2}\right) = 0, \qquad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right) = \nabla^2\chi. \tag{4.17}$$

On transforming these equations, we have
$$\{D^2+(p+2)^2\}\bar{\bar{\psi}}_1 = 0 \tag{4.18}$$

and
$$-(p+1)D\bar{\psi}_1 = (D^2+p^2)\bar{\bar{\chi}}. \tag{4.19}$$

Eliminating ψ_1, (4.7) now become
$$\left.\begin{aligned} 2\mu\int_0^\infty U_r r^p\,dr &= p\bar{\bar{\chi}}-(1-\sigma)\frac{(D^2+p^2)\bar{\bar{\chi}}}{(p+1)} \\ 2\mu\int_0^\infty U_\theta r^p\,dr &= -D\bar{\bar{\chi}}-(1-\sigma)\frac{D(D^2+p^2)\bar{\bar{\chi}}}{(p+1)(p+2)} \end{aligned}\right\}. \tag{4.20}$$

Inversion formulae. Application of (4.2) to the stresses and displacements already obtained leads to
$$\left.\begin{aligned} \Theta' &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (D^2+p^2)\bar{\bar{\chi}}r^{-p-2}\,dp \\ \Phi' &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (D+ip)\{D+i(p+2)\}\bar{\bar{\chi}}r^{-p-2}\,dp \\ D' &= -\frac{1}{4\pi\mu}\int_{c-i\infty}^{c+i\infty} (D+ip)\left[1+\frac{(1-\sigma)(D-ip)\{D-i(p+2)\}}{(p+1)(p+2)}\right]\bar{\bar{\chi}}r^{-p-1}\,dp \end{aligned}\right\} \tag{4.21}$$

The boundary conditions. These may be conditions of displacement or of stress. Using the complex boundary stress $\breve{r\theta}+i\breve{\theta\theta}$ we may write

$$\int_0^\infty (\breve{r\theta}+i\breve{\theta\theta})r^{p+1}\,dr = (p+1)(D+ip)\bar{\bar{\chi}} \tag{4.22}$$

so that if

$$\int_0^\infty (\breve{r\theta}+i\breve{\theta\theta})r^{p+1}\,dr = t(p)$$

then

$$\begin{aligned}(p+1)(D+ip)\bar{\bar{\chi}} &= t_1(p) \quad \text{on } \theta = \alpha\\ &= t_2(p) \quad \text{on } \theta = -\alpha,\end{aligned} \tag{4.23}$$

where $\theta = \pm\alpha$ are the flanks of a wedge with origin at the vertex. Similarly, if the displacements are given on the flanks and

$$\int_0^\infty (U_r+iU_\theta)r^p\,dr = d(p),$$

then

$$\begin{aligned}-\frac{i}{2\mu}(D+ip)\left[1+\frac{(1-\sigma)(D-ip)\{D-i(p+2)\}}{(p+1)(p+2)}\right]\bar{\bar{\chi}} &= d_1(p) \quad \text{on } \theta = \alpha\\ &= d_2(p) \quad \text{on } \theta = -\alpha.\end{aligned}$$

We may of course have displacements given on one boundary and stresses on the other.

5. Isolated couple nucleus at $z = d$ (real) with stress-free boundaries

Using the vertex of the wedge as origin, the complex potentials (3.3) become

$$\Omega(z) = 0, \qquad \omega(z) = \frac{2iG}{\pi}\log z_1 \quad \text{where } z_1 = z-d. \tag{5.1}$$

These give rise to flank stresses

$$\breve{r\theta}+i\breve{\theta\theta} = -\frac{G}{2\pi}\frac{1}{(r-de^{\pm i\alpha})^2} \quad \text{on } \theta = \pm\alpha. \tag{5.2}$$

To remove this stress we accordingly take

$$t_1(p) = \frac{G}{2\pi}\int_0^\infty \frac{r^{p+1}\,dr}{(r-de^{i\alpha})^2},$$

which may be evaluated by a contour integration giving

$$t_1(p) = \frac{Gd^p e^{-ip(\pi-\alpha)}(p+1)}{2\sin p\pi}, \tag{5.3}$$

also

$$t_2(p) = \overline{t_1(\bar{p})}. \tag{5.4}$$

Further, for convergence of the integral $-2 < p < 0$ which agrees with (4.15). The boundary condition (4.23) accordingly gives

$$pAe^{ip\alpha}+B(p+1)e^{i(p+2)\alpha}-\bar{B}e^{-i(p+2)\alpha} = \frac{iGd^pe^{-ip(\pi-\alpha)}}{4\sin p\pi}$$

and
$$pAe^{-ip\alpha}+B(p+1)e^{-i(p+2)\alpha}-\bar{B}e^{i(p+2)\alpha} = \frac{iGd^pe^{ip(\pi-\alpha)}}{4\sin p\pi},$$

leading to

$$A = \frac{iGd^p}{4pG(p,\alpha)}\,\frac{K(p,\alpha)\sin p\pi+G(p,\alpha)\cos p\pi}{\sin p\pi} \tag{5.5}$$

and
$$B = -\frac{iGd^p}{4G(p,\alpha)}, \tag{5.6}$$

where
$$\left.\begin{aligned} G(p,\alpha) &= (p+1)\sin 2\alpha-\sin 2(p+1)\alpha \\ K(p,\alpha) &= (p+1)\cos 2\alpha+\cos 2(p+1)\alpha \end{aligned}\right\}. \tag{5.7}$$

The stresses may now be evaluated from (4.21); we find

$$\Theta' = -\frac{G}{\pi i r^2}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{d}{r}\right)^p\frac{(p+1)\sin(p+2)\theta}{G(p,\alpha)}\,dp, \tag{5.8}$$

in which the complex $p = c+i\eta$ replaces the real p of the Mellin transform and we must take $-2 < c < 0$. If we attempt to press the value of c to the limit zero, the integrand of Θ' must be examined for possible poles on the imaginary axis. The zeros of the denominator $G(p,\alpha)$ in the range of integration occur at $p = 0, -1, -2$ but only $p = 0$ is a pole. We therefore take $c = 0$ and put $p = i\eta$ in (5.8).

Writing now $$G(p,\alpha) = E_1+iO_1,$$
where

$$E_1 = \sin 2\alpha(1-\cosh 2\eta\alpha), \qquad O_1 = \eta\sin 2\alpha-\cos 2\alpha\sinh 2\eta\alpha, \tag{5.9}$$

$$\begin{aligned} E &= E_1(\sin 2\theta\cosh\eta\theta-\eta\cos 2\theta\sinh\eta\theta)+\\ &\qquad +O_1(\eta\sin 2\theta\cosh\eta\theta+\cos 2\theta\sinh\eta\theta),\\ O &= E_1(\eta\sin 2\theta\cosh\eta\theta+\cos 2\theta\sinh\eta\theta)-\\ &\qquad -O_1(\sin 2\theta\cosh\eta\theta-\eta\cos 2\theta\sinh\eta\theta), \end{aligned}$$

we have
$$\Theta' = -\frac{G}{\pi r^2}\left[\lim_{\epsilon\to 0}\left\{\int\limits_{-\infty}^{-\epsilon}+\int\limits_{\epsilon}^{\infty}\right\}\frac{e^{im\eta}(E+iO)}{E_1^2+O_1^2}\,d\eta-\pi R\right]$$

and $m = \log(d/r)$; R is the residue of the integrand at $p = 0$, so that finally

$$\Theta' = -\frac{G}{\pi r^2}\left[2\int\limits_0^{\infty}\frac{E\cos m\eta-O\sin m\eta}{E_1^2+O_1^2}\,d\eta-\frac{\pi\sin 2\theta}{\sin 2\alpha-2\alpha\cos 2\alpha}\right]. \tag{5.10}$$

Similarly Φ' may be reduced to

$$\Phi' = -\frac{G}{2\pi r^2}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{d}{r}\right)^p \frac{(p+1)e^{ip\theta}}{G(p,\alpha)\sin p\pi} Z(p,\theta)\, dp, \tag{5.11}$$

where $\quad Z(p,\theta) = G(p,\alpha)\cos p\pi + [K(p,\alpha)-(p+2)e^{2i\theta}]\sin p\pi. \quad$ (5.12)

Since $Z(-2,\theta) = 0$, $p = 0$ is again the only pole of the integrand. Writing

$$\begin{aligned}
E_2 &= \cos 2\alpha(1+\cosh 2\eta\alpha),\\
O_2 &= \eta\cos 2\alpha - \sin 2\alpha \sinh 2\eta\alpha,\\
E_3 &= -O_2\sinh \pi\eta + E_1\cosh \pi\eta + \eta e^{2i\theta}\sinh \pi\eta,\\
O_3 &= E_2\sinh \pi\eta + O_1\cosh \pi\eta - 2e^{2i\theta}\sinh \pi\eta,\\
E_4 &= E_1 E_3 + O_1 O_3 + \eta(O_1 E_3 - E_1 O_3),\\
O_4 &= E_1 O_3 - O_1 E_3 + \eta(E_1 E_3 + O_1 O_3),
\end{aligned}$$

we have finally

$$\Phi' = -\frac{Gi}{2\pi r^2}\left[2\int\limits_0^\infty \frac{O_4\cos k\eta + E_4\sin k\eta}{\sinh \pi\eta(E_1^2+O_1^2)}\, d\eta - \pi R_1\right], \tag{5.13}$$

where $\quad k = m+i\theta \quad$ and $\quad \pi R_1 = 1+\dfrac{2\pi(\cos 2\alpha - e^{2i\theta})}{\sin 2\alpha - 2\alpha\cos 2\alpha}. \quad$ (5.14)

The displacement is given by (4.21) as follows:

$$(D-ip)(D+ip)\{D-i(p+2)\}\bar{\bar{\chi}} = 8i(p+1)(p+2)\bar{B}e^{-i(p+2)\theta},$$

leading to $\quad U_r + iU_\theta = \dfrac{G}{8\pi\mu r}\displaystyle\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{d}{r}\right)^p \frac{Y(p,\theta)e^{ip\theta}}{\sin p\pi\, G(p,\alpha)}\, dp, \quad$ (5.15)

where

$$Y(p,\theta) = \sin p\pi[\{K(p,\alpha)-(p+1)e^{2i\theta}\}+(3-4\sigma)e^{-2i(p+1)\theta}] + G(p,\alpha)\cos p\pi, \tag{5.16}$$

so that, writing

$$\begin{aligned}
E_5 &= E_1\cosh \pi\eta + \sinh \pi\eta[\eta e^{2i\theta} - O_2 + (3-4\sigma)ie^{-2i\theta}\sinh 2\eta\theta],\\
O_5 &= O_1\cosh \pi\eta + \sinh \pi\eta[E_2 - e^{2i\theta} + (3-4\sigma)e^{-2i\theta}\cosh 2\eta\theta],\\
E_6 &= E_1 E_5 + O_1 O_5, \qquad O_6 = O_5 E_1 - E_5 O_1,
\end{aligned}$$

we have

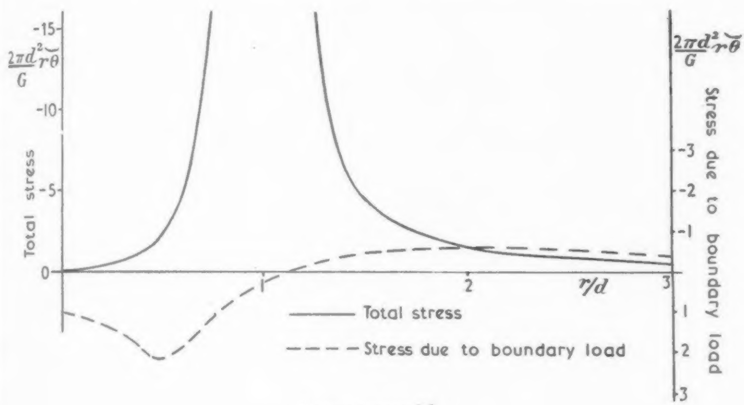
$$U_r + iU_\theta = \frac{Gi}{8\pi\mu r}\left[2\int\limits_0^\infty \frac{E_6\sin k\eta + O_6\cos k\eta}{\sinh \pi\eta(E_1^2+O_1^2)}\, d\eta - \pi R_2\right], \tag{5.17}$$

where $\quad \pi R_2 = 1+\dfrac{\pi[2\cos 2\alpha - e^{2i\theta} + (3-4\sigma)e^{-2i\theta}]}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$

The complete solution is now obtained formally by adding to the stresses and displacements of (5.10), (5.13), (5.17) those due to the original complex potentials as given in (3.4) and (3.5).

Evaluation of these integrals does not appear possible for a general value of α and recourse must be made to numerical computation. The work involved is of a very elaborate character, but calculation of the shear stress (the only non-zero component) on the axis of symmetry has been made from (5.13) for the case $\alpha = \frac{1}{4}\pi$; the values are given in the following table.

r/d	·334	·504	·663	1·000	1·505	1·985	2·994
$\frac{2\pi d^2}{G}\widetilde{r\theta}$	1·66	2·21	1·66	·32	−·46	−·57	−·42



Shear stress $\widetilde{r\theta}$.

6. Semi-infinite plate under couple nucleus

An algebraic solution of the above problem can be found for the case of a semi-infinite plate corresponding to the value $\alpha = \frac{1}{2}\pi$ since the integrals may then be integrated directly. This accounts for the fact that explicit integrated forms for the complex potentials cannot be found in general but have been found for this special case by Stevenson (**2**). It is of interest to see how the above results reduce to and agree with these.

From (5.10) we have

$$-\frac{\pi r^2}{2G}\Theta' + \tfrac{1}{2}\sin 2\theta = I_1 \sin 2\theta + I_2 \cos 2\theta,$$

where
$$I_1 = \int_0^\infty \frac{\cosh \eta\theta \sin m\eta + \eta \cosh \eta\theta \cos m\eta}{\sinh \pi\eta}\, d\eta,$$

$$I_2 = \int_0^\infty \frac{\sinh \eta\theta \cos m\eta - \eta \sinh \eta\theta \sin m\eta}{\sinh \pi\eta}\, d\eta.$$

The values of these integrals are given in (**6**, p. 277) and lead to

$$\Theta' = \frac{4Gr\sin\theta(d+r\cos\theta)}{\pi(r^2+d^2+2rd\cos\theta)^2}, \tag{6.1}$$

$$\Phi' = \frac{Gi}{\pi}\,\frac{de^{-2i\theta}-r(e^{-3i\theta}+2e^{-i\theta})}{(d+re^{-i\theta})^3}, \tag{6.2}$$

$$D' = \frac{Gi}{4\pi\mu}\left[\frac{r(1+e^{-2i\theta})}{(d+re^{-i\theta})^2}-\frac{\kappa e^{-2i\theta}}{(r+de^{-i\theta})}\right]. \tag{6.3}$$

Stevenson shows that the boundary stresses may be removed in this case by the complex potentials

$$\Omega(z) = -\frac{2iG}{\pi}\frac{1}{z_2}, \qquad \omega(z) = \frac{2iG}{\pi}\left\{\log z_2+\frac{d}{z_2}\right\}, \qquad z_2 = z+d, \tag{6.4}$$

the stresses and displacements from which agree with the above results.

7. Isolated couple nucleus at $z = d$ (real) with rigid boundaries

Using the complex potentials (5.1) the displacement of the flanks is

$$U_r+iU_\theta = \frac{iG}{4\pi\mu}\,\frac{1}{r-de^{\pm i\alpha}} \quad \text{on } \theta = \pm\alpha. \tag{7.1}$$

To remove this displacement we accordingly take

$$d_1(p) = -\frac{iG}{4\pi\mu}\int\limits_0^\infty \frac{r^p\,dr}{r-de^{i\alpha}}, \tag{7.2}$$

which may be evaluated by a contour integration giving

$$d_1(p) = \frac{iGd^pe^{-ip(\pi-\alpha)}}{4\mu\sin p\pi}, \tag{7.3}$$

also we must have $$d_2(p) = -\overline{d_1(p)}, \tag{7.4}$$

and for convergence of the integral

$$-1 < p < 0. \tag{7.5}$$

The boundary condition now becomes

$$pA+B(p+1)e^{2i\alpha}+\kappa\bar{B}e^{-i(2p+2)\alpha} = \frac{iG}{4}\,\frac{d^pe^{-ip\pi}}{\sin p\pi},$$

and $$pA+B(p+1)e^{-2i\alpha}+\kappa\bar{B}e^{i(2p+2)\alpha} = \frac{iG}{4}\,\frac{d^pe^{ip\pi}}{\sin p\pi},$$

leading to

$$A = \frac{iGd^p}{4pH_1(p,\alpha)\sin p\pi}[H_1(p,\alpha)\cos p\pi+J_1(p,\alpha)\sin p\pi], \tag{7.6}$$

$$B = -\frac{iGd^p}{4H_1(p,\alpha)}, \tag{7.7}$$

where
$$\left.\begin{aligned} H_1(p,\alpha) &= (p+1)\sin 2\alpha+\kappa\sin 2(p+1)\alpha \\ J_1(p,\alpha) &= (p+1)\cos 2\alpha-\kappa\cos 2(p+1)\alpha \end{aligned}\right\}. \tag{7.8}$$

The stresses may now be obtained from (4.21) as follows:

$$\Theta' = -\frac{G}{\pi i r^2}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{d}{r}\right)^p \frac{(p+1)\sin(p+2)\theta}{H_1(p,\alpha)}\,dp, \tag{7.9}$$

where $-1 < c < 0$.

In this case the integrand has no poles in the range since $\kappa \neq 1$ and so we take $c = 0$ and put $p = i\eta$. Writing

$$H_1(p,\alpha) = X_1+iY_1,$$

where

$$X_1 = \sin 2\alpha(1+\kappa\cosh 2\eta\alpha); \qquad Y_1 = \eta\sin 2\alpha+\kappa\cos 2\alpha\sinh 2\eta\alpha, \tag{7.10}$$

$$E_7 = X_1(\sin 2\theta\cosh\eta\theta - \eta\cos 2\theta\sinh\eta\theta)+ \\ +Y_1(\eta\sin 2\theta\cosh\eta\theta+\cos 2\theta\sinh\eta\theta),$$

$$O_7 = X_1(\eta\sin 2\theta\cosh\eta\theta+\cos 2\theta\sinh\eta\theta)- \\ -Y_1(\sin 2\theta\cosh\eta\theta-\eta\cos 2\theta\sinh\eta\theta),$$

we have
$$\Theta' = -\frac{2G}{\pi r^2}\int\limits_0^\infty \frac{E_7\cos m\eta-O_7\sin m\eta}{X_1^2+Y_1^2}\,d\eta. \tag{7.11}$$

Also if
$$J_1(p,\alpha) = X_2+iY_2,$$

where

$$X_2 = \cos 2\alpha(1-\kappa\cosh 2\eta\alpha); \qquad Y_2 = \eta\cos 2\alpha+\kappa\sin 2\alpha\sinh 2\eta\alpha,$$

$$X_3 = -Y_2\sinh\eta\pi+X_1\cosh\eta\pi+\eta e^{2i\theta}\sinh\eta\pi,$$

$$Y_3 = X_2\sinh\eta\pi+Y_1\cosh\eta\pi-2e^{2i\theta}\sinh\eta\pi,$$

$$E_8 = X_1X_3+Y_1Y_3+\eta(Y_1X_3-X_1Y_3),$$

$$O_8 = X_1Y_3-Y_1X_3+\eta(X_1X_3+Y_1Y_3),$$

then
$$\Phi' = -\frac{Gi}{2\pi r^2}\left[2\int\limits_0^\infty \frac{O_8\cos k\eta+E_8\sin k\eta}{\sinh\eta\pi(X_1^2+Y_1^2)}\,d\eta-1\right]. \tag{7.12}$$

Writing

$$X_4 = X_1 \cosh \eta\pi + \sinh \eta\pi[\eta e^{2i\theta} - Y_2 + \kappa i e^{-2i\theta} \sinh 2\eta\theta],$$
$$Y_4 = Y_1 \cosh \eta\pi + \sinh \eta\pi[X_2 - e^{2i\theta} + \kappa e^{-2i\theta} \cosh 2\eta\theta],$$
$$E_9 = X_4 X_1 + Y_4 Y_1; \qquad O_9 = Y_4 X_1 - Y_1 X_4,$$

then
$$U_r + iU_\theta = \frac{Gi}{8\pi\mu r}\left[2\int\limits_0^\infty \frac{E_9 \sin k\eta + O_9 \cos k\eta}{\sinh \pi\eta(X_1^2+Y_1^2)}\,d\eta - 1\right]. \tag{7.13}$$

Again, to complete the solution, the stresses and displacements due to the original complex potentials must be added.

8. Isolated force problems

The above theory and problems forms part of a thesis entitled 'Generalized plane stress in an elastic wedge (under distributed and isolated loads)', for the Ph.D. degree of the University of London (1953), in which isolated forces are also applied to points of the wedge.

Forces acting on the flanks at $z_0 = de^{\pm i\alpha}$ may be treated using complex potentials which include terms of type

$$\Omega(z) = E \log z_1, \qquad \omega(z) = -\bar{E} z_1 \log z_1 - \bar{z}_0 E \log z_1, \tag{8.1}$$

whilst a load at a point on the axis requires

$$\Omega(z) = -P \log z_1, \qquad \omega(z) = \kappa P z_1 \log z_1 + dP \log z_1. \tag{8.2}$$

A fundamental difference between the form of solution in these two cases arises from the intrusive κ in the $\omega(z)$ of (8.2). This has the effect of doubling the labour involved. Since the solutions to these problems are so elaborate they have not been detailed in this paper but may be referred to in the thesis mentioned above.

9. Conclusion

The facilities offered by the Mellin transform have allowed mathematical solutions to be obtained for nuclei of force or couple acting at any point of an infinite wedge, so that the usual tentative approach offered by an Airy stress-function method has been avoided. At the same time the use of the stress combinations has shortened the amount of work involved, and often, particularly in the general theory of section 3, has led to a neater expression of the results.

REFERENCES

1. A. C. STEVENSON, *Phil. Mag.* (7) **34** (1943), 766.
2. —— *Proc. Roy. Soc.* A, **184** (1945), 129.
3. C. J. TRANTER, *Quart. J. Mech. and Applied Math.* **1** (1948), 125.
4. W. M. SHEPHERD, *Proc. Roy. Soc.* A, **148** (1935), 284.
5. E. G. COKER and L. N. G. FILON, *Photoelasticity* (Cambridge, 1931), p. 168.
6. S. EDWARDS, *Integral Calculus*, vol. 2 (Macmillan, 1922).

SOME PROBLEMS OF THIN CLAMPED ELASTIC PLATES

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SUMMARY

This paper deals with problems of transverse displacements of thin elastic plates occupying the following regions: (*a*) half-planes, (*b*) those which can be mapped conformally on to a half-plane, (*c*) infinite strips, the boundaries being clamped. Solutions to plate problems have been given by several writers, Stevenson (**1**), Hopkins (**2**), and Dean (**3**) *inter alia*. The present investigation aims at general methods rather than particular solutions. Proofs of uniqueness are given, with emphasis on the difficulties in the case of materials extending to infinity in some, but not all, directions. Problems of isolated interior loading are considered for all the above types of region.

1. Fundamental equations

THROUGHOUT this paper rectangular Cartesian axes $O(x, y)$ are chosen in the mid-planes of the plates. The notation is that due to Stevenson (**1**). All relevant mean stresses and mean stress couples may be expressed in terms of w, the transverse displacement of the mid-plane, where

$$\nabla_1^4 w = f(x,y). \tag{1.1}$$

The general solution to (1.1) may be written

$$w = \bar{z}\Omega(z)+z\overline{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z})+F(z,\bar{z}), \tag{1.2}$$

where $F(z,\bar{z})$ is a particular integral of (1.1) and $\Omega(z)$, $\omega(z)$ are functions of $z = x+iy$ which are analytic in the region R of the z-plane which is occupied by elastic material. Problems of given surface loading and specified conditions of boundary clamping are considered. Thus, w, $\partial w/\partial n$, where $\partial/\partial n$ denotes differentiation along the normal to the boundary C of the material, are assumed given along C. Such information yields immediately the boundary values of $\partial w/\partial z$. The existence of solutions of such problems may be shown by methods similar to those used by Sherman (**4**).

2. Uniqueness of solutions

It is assumed that the region R is bounded by contours $C_1, C_2,..., C_n$ and a finite surrounding contour C_{n+1}. Cuts are made to make R simply connected, and the contours $C_1,..., C_{n+1}$ and the cuts, traversed once in each direction, are referred to as the contour C. Let w_1, w_2 denote two systems of transverse displacements corresponding to the same surface loading of the plate, including isolated forces, which give singularities in

w, and to the same boundary conditions. Then the function $\chi \equiv w_1 - w_2$ satisfies the equation

$$\nabla_1^4 \chi = 0 \tag{2.1}$$

in R, and, on the boundary of the material, we have

$$\chi = \frac{\partial \chi}{\partial z} = 0. \tag{2.2}$$

Further, since w_1, w_2 have the same specified isolated singularities in R, are uniform, and have uniform partial derivatives, except at these singularities, the function χ is free from singularities, is uniform, and has uniform derivatives throughout the open region defined by R.

Taking

$$\chi = \bar{z}\Omega_0(z) + z\bar{\Omega}_0(\bar{z}) + \omega_0(z) + \bar{\omega}_0(\bar{z}), \tag{2.3}$$

then

$$\frac{\partial \chi}{\partial z} = \bar{z}\Omega_0'(z) + \bar{\Omega}_0(\bar{z}) + \omega_0'(z), \tag{2.4}$$

$$\frac{\partial^2 \chi}{\partial z \partial \bar{z}} = \Omega_0'(z) + \bar{\Omega}_0'(\bar{z}), \tag{2.5}$$

$$\frac{\partial^3 \chi}{\partial z \partial \bar{z}^2} = \bar{\Omega}_0''(\bar{z}), \tag{2.6}$$

where accents denote derivatives.

The boundary conditions (2.2), together with the uniformity of the above functions, make the integral

$$I \equiv \int\limits_C \left[\frac{\partial \chi}{\partial z}\frac{\partial^2 \chi}{\partial z \partial \bar{z}}\, dz + \chi \frac{\partial^3 \chi}{\partial z \partial \bar{z}^2}\, d\bar{z}\right] \tag{2.7}$$

equal to zero. Assuming continuity of Ω_0, Ω_0', Ω_0'', ω_0, ω_0' in the closed region defined by the material, I may be transformed to a surface integral

$$I = 2i \int\limits_R \left[\left(\frac{\partial^2 \chi}{\partial z \partial \bar{z}}\right)^2 + \frac{\partial \chi}{\partial z}\frac{\partial^3 \chi}{\partial z \partial \bar{z}^2} - \frac{\partial \chi}{\partial z}\frac{\partial^3 \chi}{\partial z \partial \bar{z}^2} - \chi \frac{\partial^4 \chi}{\partial z^2 \partial \bar{z}^2}\right] dS.$$

Thus

$$\int\limits_R [\Omega_0'(z) + \bar{\Omega}_0'(\bar{z})]^2\, dS = 0, \tag{2.8}$$

so that

$$\Omega_0'(z) + \bar{\Omega}_0'(\bar{z}) = 0 \tag{2.9}$$

throughout R. Thus (**5**)

$$\Omega_0(z) = icz + \alpha, \tag{2.10}$$

where c is a real constant and α is a complex constant.

From (2.4), (2.10) it is evident that $\omega_0'(z)$ is uniform. Also $\partial\chi/\partial z = 0$ on $C_1,..., C_{n+1}$, so that

$$\omega_0'(z) = -\bar{z}ic + ic\bar{z} - \bar{\alpha} = -\bar{\alpha} \tag{2.11}$$

on the boundary. If z is any point in the interior of R,

$$\omega_0'(z) = \frac{1}{2\pi i} \int\limits_C \frac{-\bar{\alpha}\, d\zeta}{\zeta - z} = -\bar{\alpha}, \tag{2.12}$$

the use of the Cauchy integral being justified by continuity and uniformity of $\omega_0'(z)$. Thus

$$\omega_0(z) = -\bar{\alpha}z + \beta, \tag{2.13}$$

where β is a constant, throughout R. However, on $C_1,\ldots, C_{n+1}$, $\chi = 0$, so that here

$$\bar{z}(icz+\alpha)+z(-ic\bar{z}+\bar{\alpha})+\beta+\bar{\beta}+(-\bar{\alpha}z-\alpha\bar{z}) = 0.$$

Hence

$$\beta = ik, \tag{2.14}$$

where k is a real constant. Finally

$$\omega_0(z) = -\bar{\alpha}z + ik. \tag{2.15}$$

From (2.10), (2.15) it follows that

$$\chi \equiv 0, \qquad \text{i.e.} \qquad w_1 \equiv w_2 \tag{2.16}$$

throughout R.

3. Uniqueness of solution when C_{n+1} tends to infinity in all directions

Assume initially that the conditions specified at infinity imply no restrictions on Ω, ω which satisfy the conditions

$$\Omega'' = o(z^{-2}), \quad \Omega' = o(z^{-1}), \quad \Omega, \omega' = O(1), \quad \omega = O(z). \tag{3.1}$$

Then, at infinity, χ satisfies the conditions

$$\chi = O(z), \quad \frac{\partial\chi}{\partial z} = O(1), \quad \frac{\partial^2\chi}{\partial z \partial\bar{z}} = o(z^{-1}), \quad \frac{\partial^3\chi}{\partial z\partial\bar{z}^2} = o(z^{-2}). \tag{3.2}$$

Hence the integrands in (2.7) are $o(z^{-1})$ at infinity, and, since C_{n+1} may be chosen to have a length which is $O(z)$, the integral takes the value zero as C_{n+1} tends to infinity. The equations of the preceding section hold as far as (2.12), which is replaced by

$$2\pi i\omega_0'(z) = \sum_{r=1}^{n} \int\limits_{C_r} \frac{+\bar{\alpha}\, d\zeta}{\zeta - z} + \lim \int\limits_{C_{n+1}} \frac{\omega_0'(\zeta)}{\zeta - z}\, d\zeta. \tag{3.3}$$

The first integrals are zero, whilst the last integrand is $O(\zeta^{-1})$ at most. Assuming that

$$\lim_{\zeta\to\infty} \frac{\zeta\omega_0'(\zeta)}{\zeta - z} = -\bar{\alpha}', \tag{3.4}$$

a complex constant, the same in all directions, then

$$\omega_0'(z) = -\bar{\alpha}'. \tag{3.5}$$

Consideration of $\partial\chi/\partial z$ on $C_1,\ldots, C_n$ shows that $\alpha' = \alpha$. The remainder of the proof is as above.

The demands made in (3.1) may be justified as follows. Let $\mathscr{C}$ denote any large contour enclosing $C_1,\ldots, C_n$ and a large circle $\mathscr{C}_0$ which itself surrounds $C_1,\ldots, C_n$. Since $\mathscr{C}$ lies entirely within the elastic material

$$\mathrm{Cy}_{\mathscr{C}} \frac{\partial^2 w}{\partial z \partial \bar{z}} = \mathrm{Cy}_{\mathscr{C}} [\Omega'(z)+\bar{\Omega}'(\bar{z})] = 0. \tag{3.6}$$

Hence
$$\mathrm{Cy}_{\mathscr{C}} \Omega'(z) = 2\pi i \gamma, \tag{3.7}$$

where γ is a real constant.

It can be seen that
$$\mathrm{Cy}_{\mathscr{C}} \Omega''(z) = \mathrm{Cy}_{\mathscr{C}} \omega''(z) = 0. \tag{3.8}$$

Thus outside $\mathscr{C}_0$, $\Omega'(z)-\gamma \log z$, $\omega''(z)$ may be expanded in Laurent series, and it can be shown that the most general forms of Ω, ω at infinity are

$$\Omega(z) = \gamma z \log z + A \log z + B + \sum_{n=1}^{\infty} (a_n z^n + b_n z^{-n}), \tag{3.9}$$

$$\omega(z) = \bar{A} z \log z + \delta \log z + Cz + D + \sum_{n=1}^{\infty} (c_n z^{n+1} + d_n z^{-n}), \tag{3.10}$$

where γ, δ are real constants and A, B, C, D, a_n, b_n, c_n, d_n are complex constants. It is evident that, from (3.1), the conditions specified at infinity enable the determination of the constants γ, A, a_n, c_n but no others. The terms specified at infinity give all finite stress couples and infinitesimals other than those which are $o(z^{-1})$. Since C_{n+1} has length $O(z)$, the above infinitesimal terms have the unique values required for equilibrium of the material, as in the case of generalized plane stress (**5**). The smaller terms automatically take unique values, so that the solution is unique. If the material has boundary of length $O(z)$ at infinity but does not tend to infinity in all directions, then it is sufficient to specify all the terms at infinity other than those given in (3.1). That these conditions are also necessary is seen from an example given in section 9.

4. Plates in the form of half-planes

The plates are chosen to occupy the region $y \geqslant 0$ and two methods of solution for w under specified conditions of clamping along $y = 0$ are given briefly. They lead directly to methods of solution of problems of plates which can be mapped conformally on to the half-plane and to solutions of problems involving infinite strips.

Apart from a constant, the boundary conditions along $y = 0$ are determined by $(\partial w/\partial z)_{y=0}$. Taking w in the form

$$w = \bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}), \tag{4.1}$$

then

$$\frac{\partial w}{\partial z} = \bar{z}\Omega'(z)+\bar{\Omega}(\bar{z})+\omega'(z). \tag{4.2}$$

The potentials Ω_1, ω_1, where

$$\omega_1'(z) = -z\Omega_1'(z)+\Omega_1(z), \tag{4.3}$$

give displacements w_1, where

$$4\,\mathrm{re}\,\Omega_1(x) = \left(\frac{\partial w_1}{\partial x}\right)_{y=0} \equiv f(x), \quad \left(\frac{\partial w_1}{\partial y}\right)_{y=0} = 0. \tag{4.4}$$

Again, the potentials Ω_2, ω_2, where

$$\omega_2'(z) = -z\Omega_2'(z)-\Omega_2(z), \tag{4.5}$$

give displacements w_2, where

$$\left(\frac{\partial w_2}{\partial x}\right)_{y=0} = 0, \qquad 4\,\mathrm{im}\,\Omega_2(x) = \left(\frac{\partial w_2}{\partial y}\right)_{y=0} \equiv g(x). \tag{4.6}$$

Hence the problem of the half-plane with

$$2\left(\frac{\partial w}{\partial \bar{z}}\right)_{y=0} = f(x)+ig(x) = h(x), \quad \text{say}, \tag{4.7}$$

has been reduced to the determination of functions Ω_1, Ω_2 which are analytic in $y \geqslant 0$ and have specified real or imaginary parts when $y = 0$. If

$$\Omega \equiv \Omega_1+\Omega_2, \qquad \omega \equiv \omega_1+\omega_2, \tag{4.8}$$

then

$$\omega'(z) = -z\Omega'(z)+\Omega_1(z)-\Omega_2(z). \tag{4.9}$$

If $f(x)$, $g(x)$ can be expressed as Fourier integrals, then (**6**)

$$4\Omega(z) = H(z), \qquad 4\omega'(z) = -zH'(z)+H^*(z), \tag{4.10}$$

where

$$H(z) = \frac{1}{\pi}\int\limits_0^\infty e^{izu}\,du \int\limits_{-\infty}^\infty h(t)e^{-iut}\,dt = \frac{i}{\pi}\int\limits_{-\infty}^\infty \frac{h(t)\,dt}{z-t}, \tag{4.11}$$

and $H^*(z)$ is the transform of $\bar{h}(t)$. The Cauchy integral is the simpler for direct evaluation, but the Fourier form is more suitable when a second boundary exists in the finite part of the half-plane.

5. Isolated interior loading of the half-plane

The potentials

$$\left.\begin{aligned} \Omega_a(z) = \sum_{r=0}^{n} A_r(z-z_r)\log(z-z_r), \quad \omega_a(z) = \sum_{r=0}^{n} -\bar{z}_r A_r(z-z_r)\log(z-z_r) \\ A_r = -W_r/16\pi D \end{aligned}\right\}, \tag{5.1}$$

where D is Love's plate constant and W_r are real constants, correspond to isolated loads W_r at z_r. Let z_r be in the region $y \geqslant 0$. If the plate is clamped so that

$$\left(\frac{\partial w}{\partial z}\right)_{y=0} = 0, \tag{5.2}$$

it is clear that additional potentials free from singularities in $y \geqslant 0$ are required to counteract the effect of (5.1) along $y = 0$. The logarithmic terms given by (5.1) in the boundary condition may be removed by the addition of the 'image potentials',

$$\Omega_b(z) = \sum_{r=0}^{n}\{-A_r(z-z_r)\log(z-\bar{z}_r)\}, \quad \omega_b(z) = \sum_{r=0}^{n}\{+\bar{z}_r A_r(z-z_r)\log(z-\bar{z}_r)\}. \tag{5.3}$$

Thus
$$\Omega_0 \equiv \Omega_a+\Omega_b = \sum_{r=0}^{n} A_r(z-z_r)\log\frac{(z-z_r)}{(z-\bar{z}_r)}, \tag{5.4}$$

$$\omega_0 \equiv \omega_a+\omega_b = \sum_{r=0}^{n}\left\{-\bar{z}_r A_r(z-z_r)\log\frac{z-z_r}{z-\bar{z}_r}\right\}. \tag{5.5}$$

If w_0 is the displacement given by Ω_0, ω_0,

$$\frac{\partial w_0}{\partial z} = \sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r)\log\frac{(z-z_r)(\bar{z}-\bar{z}_r)}{(z-\bar{z}_r)(\bar{z}-z_r)} + I(z,\bar{z}), \tag{5.6}$$

where
$$I(z,\bar{z}) = \sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r)(z_r-\bar{z}_r)/(z-\bar{z}_r). \tag{5.7}$$

Thus
$$\left(\frac{\partial w_0}{\partial z}\right)_{y=0} = I(x,x) \equiv J(x) = \sum_{r=0}^{n} A_r(z_r-\bar{z}_r), \tag{5.8}$$

and

$$\left(\frac{\partial w_0}{\partial x}\right)_{y=0} = 2\,\mathrm{re}\,J(x) \equiv -f(x), \qquad \left(\frac{\partial w_0}{\partial y}\right)_{y=0} = -2\,\mathrm{im}\,J(x) \equiv -g(x). \tag{5.9}$$

The solution may be completed by the method of the previous section. Equations (4.4), (4.6) are satisfied by

$$2\Omega_1(z) = -J(z) = -2\Omega_2(z). \tag{5.10}$$

Hence, using (4.3), (4.5), the potentials additional to Ω_a, Ω_b, ω_a, ω_b are

$$\Omega_c \equiv \Omega_1+\Omega_2 = 0, \qquad \omega_c \equiv \omega_1+\omega_2 = -\int J(z)\,dz. \tag{5.11}$$

If $A_0 = A$, $A_1 = A_2 = A_3 = \ldots = A_n = 0$, the complete solution is given by the potentials

$$\Omega = A(z-z_0)\log[(z-z_0)/(z-\bar{z}_0)], \tag{5.12}$$

$$\omega = -A\bar{z}_0(z-z_0)\log[(z-z_0)/(z-\bar{z}_0)]+(\bar{z}_0-z_0)Az. \tag{5.13}$$

6. Solution by conformal mapping on to a half-plane

Let elastic material occupy a region R of the z-plane, bounded by a contour C, and let the function $z(\zeta)$, where $\zeta = \xi+i\eta$, map R conformally on to the region $\eta \geqslant 0$ of the ζ-plane, the boundary C corresponding to the axis $\eta = 0$. The boundary conditions along C are given, apart from a constant, by $(\partial w/\partial z)_{\eta=0}$. From (4.2)

$$\frac{\partial w}{\partial z} = \bar{z}(\bar{\zeta})\Omega'(\zeta)/z'(\zeta)+\bar{\Omega}(\bar{\zeta})+\omega'(\zeta)/z'(\zeta). \tag{6.1}$$

Let the required boundary conditions be

$$2\left(\frac{\partial w}{\partial z}\right)_{\eta=0} = \left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)_{\eta=0} = f(\xi)-ig(\xi) \equiv \bar{h}(\xi). \tag{6.2}$$

If

$$\omega_1'(\zeta) = z'(\zeta)\Omega_1(\zeta)-\bar{z}(\zeta)\Omega_1'(\zeta), \tag{6.3}$$

then, from (6.1),

$$\left(\frac{\partial w_1}{\partial x}\right)_{\eta=0} = 4\,\mathrm{re}\,\Omega_1(\xi), \qquad \left(\frac{\partial w_1}{\partial y}\right)_{\eta=0} = 0. \tag{6.4}$$

If

$$\omega_2'(\zeta) = -z'(\zeta)\Omega_2(\zeta)-\bar{z}(\zeta)\Omega_2'(\zeta), \tag{6.5}$$

then

$$\left(\frac{\partial w_2}{\partial x}\right)_{\eta=0} = 0, \qquad \left(\frac{\partial w_2}{\partial y}\right)_{\eta=0} = 4\,\mathrm{im}\,\Omega_2(\xi). \tag{6.6}$$

Thus the boundary conditions (6.2) are satisfied by choosing $\Omega = \Omega_1+\Omega_2$ and $\omega = \omega_1+\omega_2$, where

$$4\,\mathrm{re}\,\Omega_1(\xi) = f(\xi), \qquad 4\,\mathrm{im}\,\Omega_2(\xi) = g(\xi), \tag{6.7}$$

and

$$\omega'(\zeta) = z'(\zeta)(\Omega_1-\Omega_2)-\bar{z}(\zeta)(\Omega_1'+\Omega_2'). \tag{6.8}$$

If $f(\xi)$, $g(\xi)$ satisfy the conditions of Fourier's integral theorem, then the solution is given by

$$4\Omega(\zeta) = H(\zeta), \qquad 4\omega'(\zeta) = z'(\zeta)H^*(\zeta)-\bar{z}(\zeta)H'(\zeta), \tag{6.9}$$

where $H(\zeta)$ is obtained by replacing z by ζ in (4.11).

7. Isolated interior loading

Let the required isolated loading be given by the potentials Ω_a, ω_a given in (5.1), the points z_r lying in the region $\eta > 0$. For the corresponding displacements w_a

$$\frac{\partial w_a}{\partial z} = \sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r)\log[(z-z_r)(\bar{z}-\bar{z}_r)]+\sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r). \tag{7.1}$$

Also

$$z-z_r = z(\zeta)-z(\zeta_r) = (\zeta-\zeta_r)\phi_r(\zeta), \tag{7.2}$$

where $\phi_r(\zeta)$ are analytic and non-zero in $\eta \geqslant 0$ since $z'(\zeta)$ does not vanish in $\eta \geqslant 0$.

Thus

$$\Omega_a(\zeta) = \sum_{r=0}^{n} A_r(z-z_r)\log[(\zeta-\zeta_r)\phi_r(\zeta)], \tag{7.3}$$

$$\omega_a(\zeta) = \sum_{r=0}^{n} \{-\bar{z}(\bar{\zeta}_r)A_r(z-z_r)\log[(\zeta-\zeta_r)\phi_r(\zeta)]\}, \tag{7.4}$$

whilst (7.1) gives

$$\frac{\partial w_a}{\partial z} = \sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r)\log[(\zeta-\zeta_r)(\bar{\zeta}-\bar{\zeta}_r)\phi_r(\zeta)\bar{\phi}_r(\bar{\zeta})] + \sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r). \tag{7.5}$$

The most convenient image potentials in this case are

$$\Omega_b(\zeta) = \sum_{r=0}^{n} \{-A_r(z-z_r)\log[(\zeta-\bar{\zeta}_r)\phi_r(\zeta)]\}, \tag{7.6}$$

$$\omega_b(\zeta) = \sum_{r=0}^{n} \{+A_r\,\bar{z}(\bar{\zeta}_r)(z-z_r)\log[(\zeta-\bar{\zeta}_r)\phi_r(\zeta)]\}, \tag{7.7}$$

which are free from singularities in $\eta \geqslant 0$ and give

$$\frac{\partial w_b}{\partial z} = \sum_{r=0}^{n} A_r(\bar{z}_r-\bar{z})\log[(\zeta-\bar{\zeta}_r)(\bar{\zeta}-\zeta_r)\phi_r(\zeta)\bar{\phi}_r(\bar{\zeta})] + \\ + \sum_{r=0}^{n} \frac{A_r(z-z_r)(\bar{z}-\bar{z}_r)}{(\bar{\zeta}_r-\zeta)\phi_r(\zeta)} \frac{d}{dz}[(\zeta-\bar{\zeta}_r)\phi_r(\zeta)]. \tag{7.8}$$

For the combined potentials

$$\Omega_0 \equiv \Omega_a+\Omega_b = \sum_{r=0}^{n} \{+A_r(z-z_r)\log[(\zeta-\zeta_r)/(\zeta-\bar{\zeta}_r)]\}, \tag{7.9}$$

$$\omega_0 \equiv \omega_a+\omega_b = \sum_{r=0}^{n} \{-A_r\,\bar{z}(\bar{\zeta}_r)(z-z_r)\log[(\zeta-\zeta_r)/(\zeta-\bar{\zeta}_r)]\}, \tag{7.10}$$

$$\frac{\partial w_0}{\partial z} = \sum_{r=0}^{n} A_r(\bar{z}-\bar{z}_r)\log\frac{(\zeta-\zeta_r)(\bar{\zeta}-\bar{\zeta}_r)}{(\zeta-\bar{\zeta}_r)(\bar{\zeta}-\zeta_r)} + I(\zeta,\bar{\zeta}), \tag{7.11}$$

where
$$I(\zeta,\bar{\zeta}) = \sum_{r=0}^{n} A_r(\zeta_r-\bar{\zeta}_r)\bar{\phi}_r(\bar{\zeta})\frac{(\bar{\zeta}-\bar{\zeta}_r)}{(\zeta-\bar{\zeta}_r)}\frac{\phi_r(\zeta)}{z'(\zeta)}. \tag{7.12}$$

Thus
$$\left(\frac{\partial w_0}{\partial z}\right)_{\eta=0} = I(\xi,\xi) \equiv J(\xi), \tag{7.13}$$

where
$$J(\xi) = \sum_{r=0}^{n} \{+A_r(\zeta_r-\bar{\zeta}_r)\bar{\phi}_r(\xi)\phi_r(\xi)/z'(\xi)\} \tag{7.14}$$

$$\equiv -[f(\xi)-ig(\xi)]/2. \tag{7.15}$$

The logarithmic terms required to give the isolated loading do not

appear in the boundary function $J(\xi)$. To complete the solution the methods of the previous section may be followed, analytic functions being chosen to satisfy (6.7). Completion of the solution is exceptionally easy if $J(\zeta)$, i.e. $I(\zeta,\zeta)$, is free from singularities in $\eta \geqslant 0$; for, in such cases, $f(\xi)$, $-g(\xi)$ are the real and imaginary parts, when $\eta = 0$, of the same analytic function, viz. $-2J(\zeta)$. Hence (6.7) are satisfied by

$$2\Omega_1(\zeta) = -J(\zeta) = -2\Omega_2(\zeta). \tag{7.16}$$

Thus the required potentials are

$$\Omega_c = 0, \qquad \omega_c = -\int z'(\zeta)J(\zeta)\,d\zeta. \tag{7.17}$$

In general, however, the functions $\bar{\phi}_r(\zeta)$, and therefore $J(\zeta)$, are not suitable in $\eta \geqslant 0$ and $f(\xi)$, $-g(\xi)$ are not usually the real and imaginary parts of the same function of ζ, analytic in $\eta \geqslant 0$.

8. Isolated load in plate with clamped parabolic notched boundary

If

$$(\zeta+ia)^2 = 2iz, \tag{8.1}$$

then $\eta = 0$ corresponds to the parabola $a^4-2a^2y-x^2 = 0$ (**7**). Let there be a single isolated load $W = W_0$ at $z = z_0$ corresponding to

$$\zeta = \zeta_0 = \alpha+i\beta \quad (\beta > 0).$$

In this case

$$z-z_0 = (\zeta-\zeta_0)(\zeta+\zeta_0+2ia)/2i, \tag{8.2}$$

so that

$$\phi_0(\zeta) = (\zeta+\zeta_0+2ia)/2i. \tag{8.3}$$

The potentials Ω_0, ω_0 in (7.9), (7.10) become, replacing A_0 by A,

$$\Omega_0 = -\tfrac{1}{2}iA\,(\zeta-\zeta_0)(\zeta+\zeta_0+2ia)\log\frac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0}, \qquad \omega_0 = -\tfrac{1}{2}i\,(\bar{\zeta}_0-ia)^2\Omega_0. \tag{8.4}$$

Also,

$$J(\zeta) = -A\beta(\zeta+\bar{\zeta}_0-2ia)(\zeta+\zeta_0+2ia)/2(\zeta+ia), \tag{8.5}$$

and is free from singularities in $\eta \geqslant 0$. Hence the additional potentials are given by (7.17). In the present case

$$\omega_c(\zeta) = -iA\beta[\zeta^3+3\alpha\zeta^2+3\zeta(\zeta_0+2ia)(\bar{\zeta}_0-2ia)]/6. \tag{8.6}$$

9. Uniqueness of solution

Consider the transformation

$$(\zeta+ia)^n = 2iz, \tag{9.1}$$

where n is a positive integer. The above method of solution for a plate bounded by $\eta = 0$ may be applied without difficulty. If, however, in (6.5),

$$\Omega_2(\zeta) = \zeta, \tag{9.2}$$

then

$$\left(\frac{\partial w_2}{\partial z}\right)_{\eta=0} = 0, \tag{9.3}$$

so that, apart from a possible constant, the clamping conditions along $\eta = 0$ are not affected by the potential in (9.2) and ω_2 given by (6.5), viz.

$$\omega_2 = i\zeta(\zeta+ia)^n/2 - i(\zeta+ia)^{n+1}/2(n+1) - i(\zeta-ia)^{n+1}/2(n+1). \quad (9.4)$$

Thus $$\omega_2 = i\zeta^{n+1}(n-1)/2(n+1) + O(\zeta^n) \text{ at infinity.} \quad (9.5)$$

Hence, at infinity,

$$\Omega_2 = O(\zeta) = O(z^{1/n}), \qquad \omega_2 = O(\zeta^{n+1}) = O(z^{(n+1)/n}). \quad (9.6)$$

Since n may be made indefinitely large, it is clear that the requirements

$$\Omega = O(1), \qquad \omega = O(z) \quad (9.7)$$

made in (3.1) cannot, in general, be reduced. The solution given in the previous section is not unique, for, at infinity, the potentials $\Omega = \Omega_0 + \Omega_c$, $\omega = \omega_0 + \omega_c$ may be written

$$\Omega = -A\beta\zeta + O(1), \qquad \omega = -iA\beta\zeta^3/6 + O(\zeta^2), \quad (9.8)$$

and when $n = 2$, (9.2), (9.4) give

$$\Omega_d = \zeta, \qquad \omega_d = (i\zeta^3 - 6a\zeta^2 + 3ia^2\zeta)/6, \quad (9.9)$$

which are the same orders of magnitude as (9.8). If (9.9) is multiplied by $A\beta$ and added to (9.8), the unique solution

$$\Omega(\zeta) = -\tfrac{1}{2}iA(\zeta-\zeta_0)(\zeta+\zeta_0+2i\alpha)\log[(\zeta-\zeta_0)/(\zeta-\bar{\zeta}_0)] + A\beta\zeta, \quad (9.10)$$

$$\omega(\zeta) = -\tfrac{1}{4}A(\bar{\zeta}_0 - ia)^2(\zeta-\zeta_0)(\zeta+\zeta_0+2ia)\log[(\zeta-\zeta_0)/(\zeta-\bar{\zeta}_0)] - \\ -\zeta^2 A\beta(i\alpha+2a)/2 - iA\beta\zeta[(\zeta_0+2ia)(\zeta_0-2ia)-a^2]/2, \quad (9.11)$$

is obtained, where $\Omega = O(1)$, $\omega = O(z)$ at infinity.

10. Modified treatment of the half-plane

Throughout this paper the principal aim has been to consider interior forces. This has not hitherto been done for infinite strips, and, in order to consider problems involving these, a modified treatment of the half-plane is necessary. The previous method, which was the most suitable for extension to conformal mappings, leads to difficulties with regard to convergence of integrals when applied to infinite strips. The transverse displacement w is given by

$$w = \bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z}), \quad (10.1)$$

so that $$\frac{\partial w}{\partial z} = \frac{1}{2}\left[\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right] = \bar{z}\Omega'(z) + \bar{\Omega}(\bar{z}) + \omega'(z). \quad (10.2)$$

If $$\omega_1(z) = -z\Omega_1(z) + 2\int \Omega_1(z)\,dz, \quad (10.3)$$

then $$w_{1,y=0} = 4\,\mathrm{re}\int \Omega_1(x)\,dx, \qquad \left(\frac{\partial w_1}{\partial y}\right)_{y=0} = 0. \quad (10.4)$$

If $$\omega_2(z) = -z\Omega_2(z), \quad (10.5)$$

then $$w_{2,y=0} = 0, \qquad \left(\frac{\partial w_2}{\partial y}\right)_{y=0} = 4\,\mathrm{im}\,\Omega_2(x). \tag{10.6}$$

Hence the boundary conditions

$$w_{y=0} = k(x), \qquad \left(\frac{\partial w}{\partial y}\right)_{y=0} = g(x) \tag{10.7}$$

may be obtained by choosing

$$4\,\mathrm{re}\int \Omega_1(x)\,dx = k(x), \qquad 4\,\mathrm{im}\,\Omega_2(x) = g(x). \tag{10.8}$$

With the notation and restrictions of (4.11), the transverse displacements of the plate are given by

$$w = \mathrm{re}\{K(z) - iy[K'(z) + iG'(z)]\}. \tag{10.9}$$

It is assumed, henceforth, that specified conditions of clamping along $y = 0$ have been obtained by the above method for the boundary $y = 0$ of an infinite strip $0 \leqslant y \leqslant a$, the appropriate potentials being denoted by Ω_1, Ω_2, ω_1, ω_2. Additional potentials are required to give specified conditions along $y = a$, without altering the conditions along $y = 0$; cf. the author's treatment of generalized plane stress (**8**).

11. Potentials which give $w_{y=0} = \left(\frac{\partial w}{\partial y}\right)_{y=0} = 0$

If $$\omega_3 = -z\Omega_3 + 2\int \Omega_3\,dz, \tag{11.1}$$

and $$\int \Omega_3\,dz = \frac{i}{\pi}\int\limits_0^\infty [(\gamma_1 + i\gamma_2)e^{izu} + (\gamma_1 - i\gamma_2)e^{-izu}]\,du, \tag{11.2}$$

where γ_1, γ_2 are real functions of the real variable u, then, from (10.4),

$$w_{3,y=0} = \left(\frac{\partial w_3}{\partial y}\right)_{y=0} = 0. \tag{11.3}$$

If $$\omega_4 = -z\Omega_4, \tag{11.4}$$

and $$\Omega_4 = \frac{1}{\pi}\int\limits_0^\infty [(\delta_1 + i\delta_2)e^{izu} + (\delta_1 - i\delta_2)e^{-izu}]\,du, \tag{11.5}$$

then (10.6) shows that w_4 satisfies the conditions given for w_3 in (11.3). The functions γ_1, γ_2, δ_1, δ_2 may now be determined to give specified

conditions along $y = a$. Using these two pairs of potentials in (10.1), (10.2) and combining the results, the following conditions are obtained along $y = a$.

$$w_{y=a} = \frac{8}{\pi}\int_0^\infty \{\cos xu[-\delta_2 as+\gamma_2(s-\lambda c)]+\sin xu[-\delta_1 as+\gamma_1(s-\lambda c)]\}\,du, \tag{11.6}$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=a} = \frac{8}{\pi}\int_0^\infty \{\cos xu[-su\lambda\gamma_2-\delta_2(s+\lambda c)]+\sin xu[-\gamma_1 su\lambda-\delta_1(s+\lambda c)]\}\,du, \tag{11.7}$$

where $\lambda = au, \quad c = \cosh\lambda, \quad s = \sinh\lambda.$

Let the conditions required along $y = a$ be

$$w_{y=a} = \frac{8}{\pi}\int_0^\infty (\sigma_1\cos xu+\sigma_2\sin xu)\,du, \tag{11.8}$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=a} = \frac{8}{\pi}\int_0^\infty (\tau_1\cos xu+\tau_2\sin xu)\,du, \tag{11.9}$$

where it should be noted that the functions $\sigma_1, \sigma_2, \tau_1, \tau_2$ allow for the effect of the potentials $\Omega_1, \Omega_2, \omega_1, \omega_2$ as well as the clamping conditions along $y = a$. From (11.6)–(11.9), we find

$$\begin{aligned}\sigma_1 &= -\delta_2 as+\gamma_2(s-\lambda c),\\ \sigma_2 &= -\delta_1 as+\gamma_1(s-\lambda c),\\ \tau_1 &= -\gamma_2 su\lambda-\delta_2(s+\lambda c),\\ \tau_2 &= -\gamma_1 su\lambda-\delta_1(s+\lambda c),\end{aligned} \tag{11.10}$$

whence

$$\begin{aligned}\gamma_1 &= \frac{(s+\lambda c)\sigma_2-as\tau_2}{s^2-\lambda^2}, \qquad \delta_1 = \frac{(\lambda c-s)\tau_2-su\lambda\sigma_2}{s^2-\lambda^2},\\ \gamma_2 &= \frac{(s+\lambda c)\sigma_1-as\tau_1}{s^2-\lambda^2}, \qquad \delta_2 = \frac{(\lambda c-s)\tau_1-su\lambda\sigma_1}{s^2-\lambda^2}.\end{aligned} \tag{11.11}$$

The derivation of (11.6), (11.7) required two differentiations with respect to z under the integral sign of the integral for Ω_3 and one for Ω_4. Validity of the operation requires that

$$\sigma_1, \sigma_2 = o(u^{-4}), \qquad \tau_1, \tau_2 = o(u^{-3}) \quad \text{at} \quad u = \infty. \tag{11.12}$$

12. Conditions at $u = 0$

Assuming only that $\sigma_1, \sigma_2, \tau_1, \tau_2$ are bounded at $u = 0$, it is evident that at $u = 0$

$$
\begin{aligned}
\gamma_1 &= \frac{3}{\lambda^3}\{\sigma_2(2+2\lambda^2/5)-a\tau_2(1+\lambda^2/30)\}+O(1),\\
\gamma_2 &= \frac{3}{\lambda^3}\{\sigma_1(2+2\lambda^2/5)-a\tau_1(1+\lambda^2/30\}+O(1),\\
\delta_1 &= \frac{3}{\lambda}\left\{\frac{\tau_2}{3}-\frac{\sigma_2}{a}\right\}+O(1),\\
\delta_2 &= \frac{3}{\lambda}\left\{\frac{\tau_1}{3}-\frac{\sigma_1}{a}\right\}+O(1).
\end{aligned} \tag{12.1}
$$

The integrands in (11.2), (11.5) at $u = 0$ are

$$
\begin{aligned}
&\gamma_1(2-z^2u^2)+i\gamma_2(2izu)+O(1),\\
&\delta_1(2-z^2u^2)+i\delta_2(2izu)+O(1),
\end{aligned} \tag{12.2}
$$

respectively. Hence some modifications are required if the solution is to be valid. The formal pairs of potentials Ω_5, ω_5 and Ω_6, ω_6, where

$$
\begin{aligned}
\omega_5 &= -z\Omega_5+2\int \Omega_5\,dz,\\
\int \Omega_5\,dz &= \frac{i}{\pi}\int\limits_0^\infty [(\gamma_1+i\gamma_2)(1+izu-z^2u^2/2)+(\gamma_1-i\gamma_2)(1-izu-z^2u^2/2)]\,du,
\end{aligned} \tag{12.3}
$$

$$
\begin{aligned}
\omega_6 &= -z\Omega_6,\\
\Omega_6 &= \frac{1}{\pi}\int\limits_0^\infty [(\delta_1+i\delta_2)+(\delta_1-i\delta_2)]\,du,
\end{aligned} \tag{12.4}
$$

give zero displacements everywhere. Hence (11.2), (11.5) may be modified to the form

$$
\begin{aligned}
\int \Omega_3\,dz = \frac{i}{\pi}\int\limits_0^\infty &[(\gamma_1+i\gamma_2)(e^{izu}-1-izu+z^2u^2/2)+\\
&+(\gamma_1-i\gamma_2)(e^{-izu}-1+izu+z^2u^2/2)]\,du,
\end{aligned} \tag{12.5}
$$

$$
\Omega_4 = \frac{1}{\pi}\int\limits_0^\infty [(\delta_1+i\delta_2)(e^{izu}-1)+(\delta_1-i\delta_2)(e^{-izu}-1)]\,du. \tag{12.6}
$$

The integrands in (12.5), (12.6) at $u = 0$ are bounded. It can easily be seen that the integrands obtained by differentiations of (12.5), (12.6) with respect to z under the integral signs are also bounded at $u = 0$.

13. Isolated load in the interior of a clamped infinite strip

From (5.12), (5.13), the potentials

$$\Omega_0(z) = A(z-z_0)\log(z-z_0)/(z-\bar{z}_0), \qquad \omega_0(z) = -\bar{z}_0\Omega_0(z)+(\bar{z}_0-z)Az, \tag{13.1}$$

correspond to an isolated load at $z = z_0$ in a semi-infinite plate $y \geqslant 0$ with $w_{y=0} = (\partial w/\partial z)_{y=0} = 0$. In the following work $z_0 = \alpha+i\beta$, $0<\beta<a$. From (10.1), (10.2), the displacements along $y = a$, given by (13.1), are

$$w_{0,y=0} = A[(x-\alpha)^2+(a-\beta)^2]\log\frac{(x-\alpha)^2+(a-\beta)^2}{(x-\alpha)^2+(a+\beta)^2}+4a\beta A = -\psi(x). \tag{13.2}$$

Also

$$\left(\frac{\partial w_0}{\partial y}\right)_{y=a} = 2A(a-\beta)\log\frac{(x-\alpha)^2+(a-\beta)^2}{(x-\alpha)^2+(a+\beta)^2}+\frac{8Aa\beta(a+\beta)}{(x-\alpha)^2+(a+\beta)^2} = -\chi(x). \tag{13.3}$$

If $\psi(x)$, $\chi(x)$ are expressed in the forms of (11.8), (11.9), it is found that

$$\begin{aligned} \sigma_1 u^3e^{au}/A\pi\cos\alpha u &= \sigma_2 u^3e^{au}/A\pi\sin\alpha u \\ &= \beta u\cosh\beta u(au+1)-\sinh\beta u(1+au+a\beta u^2), \end{aligned} \tag{13.4}$$

$$\begin{aligned} \tau_1 ue^{au}/A\pi\cos\alpha u &= \tau_2 ue^{au}/A\pi\sin\alpha u \\ &= -a\beta u\cosh\beta u+\sinh\beta u(-\beta+a+a\beta u). \end{aligned} \tag{13.5}$$

Since σ_1, σ_2, τ_1, τ_1 are $O(1)$ at $u = 0$ and are $O(e^{(\beta-a)u})$ at $u = \infty$, the solution is valid.

14. Conclusion

Whilst these problems show much in common with the corresponding problem of generalized plane stress, their interest lies in the fact that each presents its own particular difficulty. This does not seem to have been appreciated by writers such as Muskhelishvili (**9**, **10**).

REFERENCES

1. A. C. Stevenson, *Phil. Mag.* (7) **33** (1942), 639–61.
2. H. G. Hopkins, *Proc. Camb. Phil. Soc.* **46** (1949), 164–81.
3. W. R. Dean, ibid. **49** (1953), 319–26.
4. D. I. Sherman, *Prikl. Mat. i Mech.* **7** (1943), 413–20.
5. R. Tiffen, *Quart. J. Mech. and Applied Math.* **5** (1952), 237–52.
6. —— ibid. **5** (1952), 344–51.
7. —— ibid. **5** (1952), 352–9.
8. —— ibid. **6** (1953), 344–69.
9. N. I. Muskhelishvili, *Some Fundamental Problems of the Mathematical Theory of Elasticity*, 3rd ed. A.N.S.S.S.R., 1949, trans. by J. R. M. Radok (1953).
10. —— *Singular Integral Equations. Boundary Problems in Function Theory and their Application to Mathematical Physics*, 2nd ed. (Moscow, 1946), English ed. 1953.

NOTE ON THE ACCELERATION OF LIN'S PROCESS OF ITERATED PENULTIMATE REMAINDER

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SUMMARY

The method of iterated penultimate remainder, introduced by S. N. Lin for approximating to factors of a polynomial, is examined from the point of view of removing divergence or accelerating convergence. It is shown how this can be done by multiplying the proposed polynomial by a suitably determined catalytic polynomial. The procedure is numerically illustrated for a linear and a quadratic factor.

1. Introduction

THE present note arose from perusal of the note by Morris and Head (**1**) on Lin's method of iterated remainder (**2**) for approximating to a polynomial factor of a given polynomial. We had elsewhere (**3, 4**) already answered the questions raised by Morris and Head on criteria of convergence, and had even made the suggestions we now develop. These suggestions seem likely to be valuable in computation, and therefore deserve more than the summary treatment we gave them. They had reference to a device, which we shall call the method of the *catalytic multiplier*, for removing divergence and greatly accelerating the convergence of Lin's iteration.

2. General criteria of convergence

It is necessary to summarize the main result of our former investigations. Let the polynomial to be factorized be

$$f_n(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots + (-)^n a_n.$$

Let

$$d_m(x) = x^m - b_1 x^{m-1} + b_2 x^{m-2} - \ldots + (-)^m b_m$$

be a polynomial factor of $f_n(x)$, and let an approximation to $d_m(x)$ be

$$t_m(x) = x^m - (b_1+\epsilon_1)x^{m-1} + (b_2+\epsilon_2)x^{m-2} - \ldots + (-)^m (b_m+\epsilon_m).$$

Let the quotient when $f_n(x)$ is divided by $d_m(x)$ be

$$q_{n-m}(x) = x^{n-m} - c_1 x^{n-m-1} + c_2 x^{n-m-2} - \ldots + (-)^{n-m} c_{n-m}.$$

Let $\boldsymbol{\epsilon}$ denote the vector $\{\epsilon_1, \epsilon_2, \ldots \epsilon_m\}$ of small errors in the coefficients of $t_m(x)$.

Lin's procedure consists in dividing $f_n(x)$ by $t_m(x)$ as far as the *penultimate* stage, at which the remainder, which we proposed to name the *penultimate remainder* (p.r.), is of the degree m of the divisor. Dividing the p.r. by a constant, so that the leading term becomes x^m, we obtain the *reduced penultimate remainder* (r.p.r.). The r.p.r. is now taken as a fresh divisor,

and in this way penultimate remaindering is iterated, line by line, on the work-sheet. The question is whether the resulting sequence of successive divisors, really a sequence of vectors having their coefficients for elements, will converge to $d_m(x)$ and, for practical purposes, converge well.

We treated (**3**, **4**) this problem from the point of view of the linear transformation of the error-vector $\boldsymbol{\epsilon}$, the elements ϵ_i being assumed so small that their powers and products of degree higher than the first were negligible. With $d_m(x)$ we associated a canonical matrix

$$B = \begin{bmatrix} b_1 & -1 & & & \\ b_2 & & -1 & & \\ b_3 & & & -1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m-1} & & & & -1 \\ b_m & & & & \cdot \end{bmatrix}$$

familiar in matrix theory as the *companion matrix* of $d_m(x)$, its characteristic polynomial, though here transposed from the more usual form. With the r.p.r. we associated the *reduced penultimate quotient* (r.p.q.), namely

$$(-)^{n-m}c_{n-m}^{-1}\{x^{n-m}-c_1\,x^{n-m-1}+c_2\,x^{n-m-2}-\ldots+(-)^{n-m-1}c_{n-m-1}\,x\}.$$

The fundamental theorem was that the matrix which transforms the error-vector $\boldsymbol{\epsilon}$ at one stage of r.p.r. iteration into the error-vector at the next is

$$R = (-)^{n-m}c_{n-m}^{-1}\{B^{n-m}-c_1\,B^{n-m-1}+\ldots+(-)^{n-m-1}c_{n-m-1}\,B\}.$$

The condition of convergence is that if ρ be the latent root of R of largest modulus, then $|\rho| < 1$; the smaller $|\rho|$ is, the more rapid the convergence. By standard matrix theory the latent roots of R are the values assumed by the r.p.q. when x takes the values $\xi_1, \xi_2,\ldots, \xi_m$, these ξ_i being the roots of $d_m(x) = 0$. We thus have a criterion of convergence based jointly on $d_m(x)$ and on the r.p.q. Naturally such a criterion in practice can be used only with tentative approximation, since both divisor and r.p.q. emerge only *de proche en proche*; but experience has shown that it serves well, and even (**3**, **4**) provides further useful information concerning $d_m(x)$. The results of Morris and Head are equivalent to cases $m = 1, m = 2$ of the general theorem.

3. Simple linear illustration

Morris and Head, illustrating the linear case $m = 1$ by the simple example

$$f_3(x) = (x-1)(x-2)(x-3) = x^3-6x^2+11x-6,$$

point out that r.p.r. iteration converges in the case of the factors $x-1$ and

$x-3$, but not in the case of $x-2$. This is because the respective values of the r.p.q. are as follows:

$$x-1;\ \text{r.p.q.}\ (x^2-5x)/6;\ \text{for}\ x=1,\ \rho=-\tfrac{2}{3}.$$
$$x-2;\qquad (x^2-4x)/3;\qquad x=2,\ \rho=-\tfrac{4}{3}.$$
$$x-3;\qquad (x^2-3x)/2;\qquad x=3,\ \rho=0.$$

The case of $x-3$ discloses a feature not explicitly commented upon by Morris and Head, namely that since $x-3$ is also a factor of the r.p.q. we have $\rho=0$; the effect being that the r.p.r. process *obliterates the linear part of error*, leaving as residuum that higher type of convergence called *quadratic* convergence, characteristic, for example, of the Newton-Raphson iteration.

In this instance the feature is accidental. We can however—and this was in fact (**4**, 333) our suggestion—deliberately produce it as follows: Let $x-\alpha$ be the trial divisor and $f_n(x)$ be the original dividend. Now replace this dividend by $(x-h)f_n(x)$, where h has been so determined that the r.p.q. of $(x-h)f_n(x)$ with respect to the divisor $x-\alpha$ is zero when $x=\alpha$. As we have said, this can be done only approximately; but it is open to us, and often advantageous, as improved values of the desired factor emerge, to recalculate h and thus further accelerate the convergence. If the factor that interests us is not linear but quadratic, $d_2(x)$, we shall replace $f_n(x)$ by $(x^2-h_1x+h_2)f_n(x)$, where h_1 and h_2 have been so determined that the r.p.q., or the p.q. simply, of $(x^2-h_1x+h_2)f_n(x)$ shall *contain* $d_2(x)$ *as a factor*. Similarly the case of a divisor of degree m would require the introduction of a *catalytic multiplier* of degree m; but for practical reasons m is unlikely to exceed 2.

4. Theory of the catalytic multiplier

We consider the general case. Let $q_{n-m}(x)$, the quotient arising when $f_n(x)$ is divided by $d_m(x)$, be again divided by $d_m(x)$, the division being continued into negative powers as far as may be required by what follows. Let the result be

$$f_n(x)/\{d_m(x)\}^2 = x^{n-2m}-k_1x^{n-2m-1}+k_2x^{n-2m-2}-\dots.$$

We require that the p.q. of

$$\{x^m-h_1x^{m-1}+h_2x^{m-2}-\dots+(-)^m h_m\}f_n(x)$$

with respect to $d_m(x)$ shall be exactly divisible by $d_m(x)$. This p.q., being penultimate, is a polynomial having zero for constant term. It follows that in the expansion of the formal product

$$\{x^m-h_1x^{m-1}+\dots+(-)^m h_m\}(x^{n-2m}-k_1x^{n-2m-1}+k_2x^{n-2m-2}-\dots)$$

the constant term and all later terms shall vanish. Taking the first m conditions for such a property, we have

$$h_m k_{n-2m+j} + h_{m-1} k_{n-2m+j+1} + \ldots + h_1 k_{n-m+j-1} + k_{n-m+j} = 0$$

$$(j = 0, 1, 2, \ldots, m-1),$$

m simultaneous equations for $h_1, h_2, \ldots, h_m$ with a persymmetric matrix. This is the simplest practical way of expressing the conditions determining $h_1, h_2, \ldots, h_m$; for the quotient

$$f_n(x)/\{d_m(x)\}^2 = q_{n-m}(x)/\{d_m(x)\}$$

will be run off on the machine (always approximately, as we insist all through) by the usual synthetic division, and the coefficients $1, k_1, k_2, \ldots$ will be shown in a single row.

In the linear case $m = 1$ there is a single equation, yielding

$$h = -k_{n-1}/k_{n-2}.$$

In the quadratic case $m = 2$ there are two equations

$$h_2 k_{n-4} + h_1 k_{n-3} + k_{n-2} = 0,$$

$$h_2 k_{n-3} + h_1 k_{n-2} + k_{n-1} = 0$$

and it is hardly necessary to point out how, for the solutions, three isobaric determinants of second order will at once be computed from $k_{n-4}, k_{n-3}, k_{n-2}, k_{n-1}$, four consecutive coefficients visible in a row of working.

These two cases are likely to be the most valuable in practice, and will now be illustrated.

5. Numerical illustrations, linear and quadratic

The Laguerre polynomial of fourth degree is

$$L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24.$$

One of its zeros is near to $x = 4{\cdot}5$. We shall find by r.p.r. iteration, with suitable catalytic multiplier, a linear factor near to $x - 4{\cdot}5$. The work-sheet, with the convention of sign we personally favour (indicating elementary symmetric functions of roots rather than alternately-signed coefficients), appears as follows:

		1	16	72	96	24
1	4·5	1	11·5	20·25	4·8875	
1	4·5	1	7·0	**−11·25**	**55·5125**	

First, $h = 55{\cdot}5125/11{\cdot}25 = 5$ nearly. We therefore multiply $L_4(x)$ by $x-5$, and proceed with r.p.r.:

r.p.r.						p.r.	
		1	21	152	456	504	120
1	4·5	1	16·0	77·75	106·125	26·4375	120
1	4·539	1	16·461	77·28352	106·21010	26·4514	120
1	4·5366	1	16·4634	77·31210	105·26575	26·45140	120
1	4·53662						

With rapid convergence we obtain the factor $x-4{\cdot}53662$. Two things may be noticed; first, that r.p.r. iteration on $L_4(x)$ unmodified would have strongly diverged in this case; second, that though the catalytic multiplier $x-5$ is quite close to the factor $x-4{\cdot}53662$, no 'fouling' has been caused. However, this in its turn provokes the cautionary remark, that the device of the catalytic multiplier will not succeed if the first approximation to a factor is too crude. There is a certain radius of effectiveness.

As a second example, given that two roots of $L_4(x) = 0$ are not far from $x = 4{\cdot}5$, $x = 1{\cdot}75$, let us combine these in the approximate quadratic factor $x^2-6{\cdot}3x+7{\cdot}9$. Unmodified r.p.r. iteration will again be found to diverge. We proceed to determine h_1, h_2:

			1	16	72	96	24
1	6·3	7·9	1	9·7	3·0		
1	6·3	7·9	1	3·4	−26·32	138·956	

The second row shows $1 = k_0, k_1, k_2, k_3$. The equations for h_1, h_2 are

$$h_2+3{\cdot}4h_1-26{\cdot}32 = 0,$$

$$3{\cdot}4h_2-26{\cdot}32h_1+138{\cdot}956 = 0,$$

yielding $h_1 = 6{\cdot}03$, $h_2 = 5{\cdot}82$.

Taking $x^2-6x+5{\cdot}8$ as approximate catalytic multiplier, we multiply $L_4(x)$ by it and proceed with r.p.r.:

r.p.r.							p.r.		
			1	22·0	173·8	620·8	1017·6	700·8	139·2
1	6·3	7·9	1	15·7	66·99	74·733	17·5611	110·4093	139·2
1	6·287	7·927	1	15·713	67·08537	74·47733	17·5733	110·4182	139·2
1	6·2826	7·9202							

Convergence is rapid. The factor in question is actually

$$x^2-6{\cdot}2824x+7{\cdot}9198 = (x-4{\cdot}5366)(x-1{\cdot}7458).$$

REFERENCES

1. J. Morris and J. W. Head, *Quart. J. Mech. and Applied Math.* **6** (1953), 391–97.
2. Shih-Nge Lin, *J. Math. and Phys.* **20** (1941), 231–42.
3. A. C. Aitken, *Proc. Roy. Soc. Edin.* A, **63** (1950), 174–91.
4. —— ibid. **63** (1952), 326–35.

CORRIGENDUM

to the paper on 'A source in a rotating fluid' by S. N. BARUA

The published figure is inconsistent with the table in the paper. It should be contracted in the axial direction by a factor 2. When corrected for this discrepancy, the bulge will appear to be less spread out along the axial direction.

Ref. Vol. VIII, Part 1, p. 22.

CORRIGENDUM

to 'A note on a paper by Davies and Walters on "The effect of finite width of area on the rate of evaporation into a turbulent atmosphere"' by L. E. PAYNE

The η in equation (12) of this paper (Vol. VII, Part 3, pp. 283–6) should be changed to a z.

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VOLUME VIII PART 2 JUNE 1955

CONTENTS

D. N. de G. Allen and R. V. Southwell: Relaxation Methods applied to determine the Motion, in Two Dimensions, of a Viscous Fluid past a Fixed Cylinder 129

R. P. Kanwal: Rotatory and Longitudinal Oscillations of Axi-symmetric Bodies in a Viscous Fluid 146

W. T. Koiter: On the Diffusion of Load from a Stiffener into a Sheet 164

J. Martinek and G. C. K. Yeh: Sound Scattering and Transmission by Thin Elastic Rectangular Plates 179

Y. C. Fung and W. H. Wittrick: A Boundary Layer Phenomenon in the Large Deflexion of Thin Plates 191

A. S. Lodge: The Transformation to Isotropic Form of the Equilibrium Equations for a Class of Anisotropic Elastic Solids . 211

D. E. R. Godfrey: Generalized Plane Stress in an Elastic Wedge under Isolated Loads 226

R. Tiffen: Some Problems of Thin Clamped Elastic Plates . . 237

A. C. Aitken: Note on the Acceleration of Lin's Process of Iterated Penultimate Remainder 251

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NICS

1955

129

146

164

179

191

211

226

237

251

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